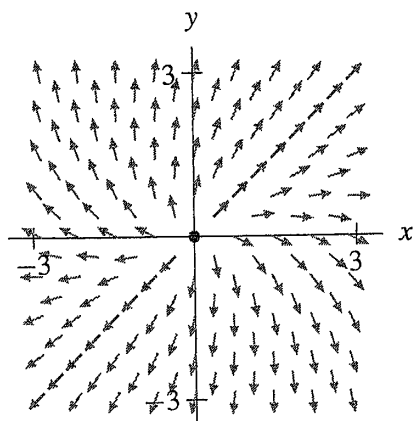
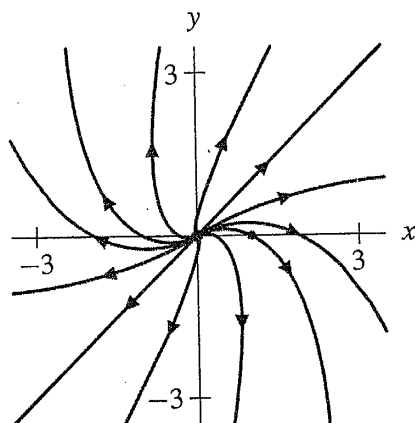


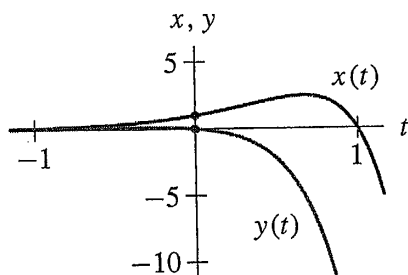
(c) Note the straight-line solutions along the line $y = x$.



(d) Since the sole eigenvalue is positive, all solutions except the equilibrium solution are unbounded as t increases. As $t \rightarrow -\infty$, the solutions with initial conditions in the half-plane $y > x$ tend to the origin tangent to the half-line $y = x$ with $y < 0$. Similarly, solutions with initial conditions in the half-plane $y < x$ tend to the origin tangent to the half-line $y = x$ with $y > 0$. Note the solution curve that goes through the initial condition $(1, 0)$.



(e) At the point $\mathbf{Y}_0 = (1, 0)$, $d\mathbf{Y}/dt = (2, -1)$. Hence, $x(t)$ is initially increasing, and $y(t)$ is initially decreasing.



③ (a) The characteristic equation is

$$(-2 - \lambda)(-4 - \lambda) + 1 = (\lambda + 3)^2 = 0,$$

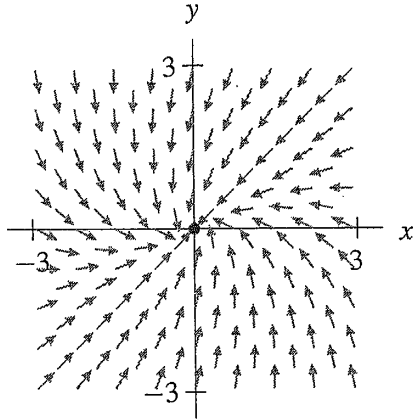
and the eigenvalue is $\lambda = -3$.

(b) To find an eigenvector, we solve the simultaneous equations

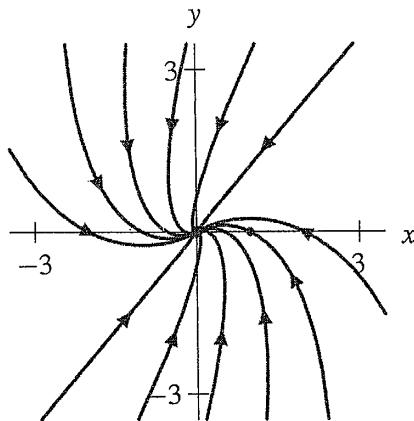
$$\begin{cases} -2x - y = -3x \\ x - 4y = -3y. \end{cases}$$

Then, $y = x$, and one eigenvector is $(1, 1)$.

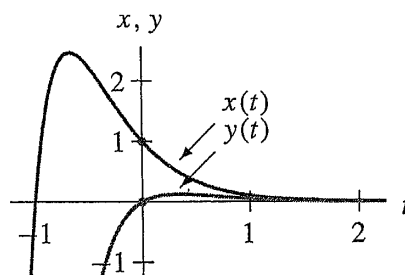
(c) Note the straight-line solutions along the line $y = x$.



(d) Since the eigenvalue is negative, any solution on the line $y = x$ tends toward the origin along $y = x$ as t increases. According to the direction field, every solution tends to the origin as t increases. The solutions with initial conditions that lie in the half-plane $y > x$ eventually approach the origin tangent to the half-line $y = x$ with $y < 0$. Similarly, the solutions with initial conditions that lie in the half-plane $y < x$ eventually approach the origin tangent to the line $y = x$ with $y > 0$.



(e) At the point $\mathbf{Y}_0 = (1, 0)$, $d\mathbf{Y}/dt = (-2, 1)$. Therefore, $x(t)$ initially decreases and $y(t)$ initially increases. The solution eventually approaches the origin tangent to the line $y = x$. Since the solution curve never crosses the line $y = x$, the graphs of $x(t)$ and $y(t)$ do not cross.



4. (a) The characteristic polynomial is

$$(-\lambda)(-2 - \lambda) + 1 = \lambda^2 + 2\lambda + 1 = (\lambda + 1)^2,$$

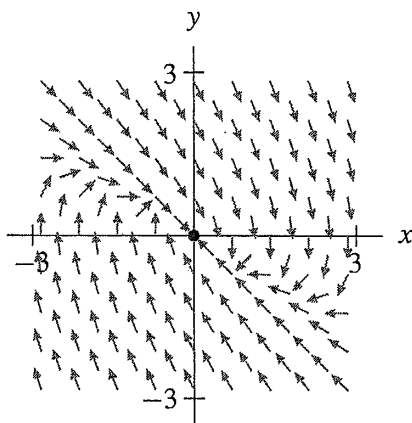
so there is only one eigenvalue, $\lambda = -1$.

- (b) To find an eigenvalue we solve

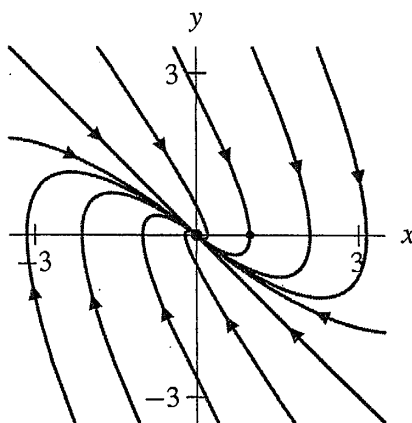
$$\begin{cases} y = -x \\ -x - 2y = -y. \end{cases}$$

These equations both simplify to $y = -x$, so $(1, -1)$ is one eigenvector.

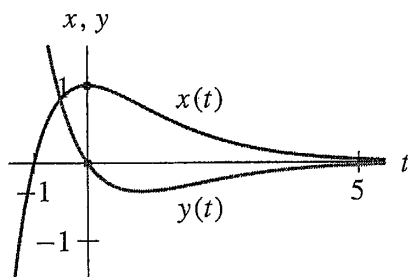
- (c) Note the straight-line solutions along the line $y = -x$.



- (d) Since the eigenvalue is negative, all solutions approach the origin as t increases. Solutions with initial conditions on the line $y = -x$ approach the origin along $y = -x$. Solutions with initial conditions that lie in the half-plane $y > -x$ approach the origin tangent to the half-line $y = -x$ with $y < 0$. Solutions with initial conditions that lie in the half-plane $y < -x$ approach the origin tangent to the half-line $y = -x$ with $y > 0$.



- (e) At the point $\mathbf{Y}_0 = (1, 0)$, $d\mathbf{Y}/dt = (0, -1)$. Therefore, $x(t)$ assumes a maximum at $t = 0$ and then decreases toward 0. Also, $y(t)$ becomes negative. Then, it assumes a (negative) minimum, and finally it is asymptotic to 0 without crossing $y = 0$.



5. (a) According to Exercise 1, there is one eigenvalue, -3 , with eigenvectors of the form $(0, y_0)$, where $y_0 \neq 0$.

To find the general solution, we start with an arbitrary initial condition $\mathbf{V}_0 = (x_0, y_0)$. Then

$$\begin{aligned} \mathbf{V}_1 &= \left[\begin{pmatrix} -3 & 0 \\ 1 & -3 \end{pmatrix} + 3 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \mathbf{V}_0 \\ &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ x_0 \end{pmatrix}. \end{aligned}$$

We obtain the general solution

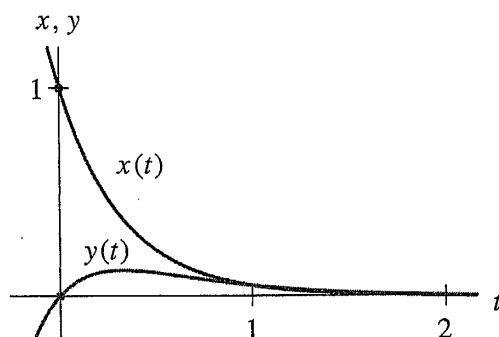
$$\mathbf{Y}(t) = e^{-3t} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + t e^{-3t} \begin{pmatrix} 0 \\ x_0 \end{pmatrix}.$$

- (b) The solution that satisfies the initial condition $(x_0, y_0) = (1, 0)$ is

$$\mathbf{Y}(t) = e^{-3t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + t e^{-3t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Hence, $x(t) = e^{-3t}$ and $y(t) = t e^{-3t}$.

- (c) Compare the graphs of $x(t) = e^{-3t}$ and $y(t) = t e^{-3t}$ with the sketches obtained in part (e) of Exercise 1.

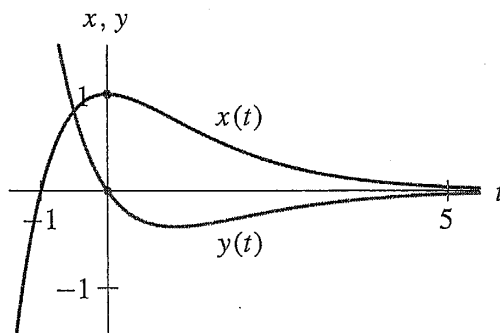


(b) The solution that satisfies the initial condition $(x_0, y_0) = (1, 0)$ is

$$\mathbf{Y}(t) = e^{-t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + te^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Hence, $x(t) = e^{-t}(t + 1)$ and $y(t) = -te^{-t}$.

(c) Compare the graphs of $x(t) = e^{-t}(t + 1)$ and $y(t) = -te^{-t}$ with those obtained in part (e) of Exercise 4.



(9) (a) By solving the quadratic equation, we obtain

$$\lambda = \frac{-\alpha \pm \sqrt{\alpha^2 - 4\beta}}{2}.$$

Therefore, for the quadratic to have a double root, we must have

$$\alpha^2 - 4\beta = 0.$$

(b) If zero is a root, we set $\lambda = 0$ in $\lambda^2 + \alpha\lambda + \beta = 0$, and we obtain $\beta = 0$.

(10) (a) To compute the limit of $te^{\lambda t}$ as $t \rightarrow \infty$ if $\lambda > 0$, we note that both t and $e^{\lambda t}$ go to infinity as t goes to infinity. So $te^{\lambda t}$ blows up as t tends to infinity, and the limit does not exist.

(b) To compute the limit of $te^{\lambda t}$ as $t \rightarrow \infty$ if $\lambda < 0$, we write

$$\lim_{t \rightarrow \infty} te^{\lambda t} = \lim_{t \rightarrow \infty} \frac{t}{e^{-\lambda t}} = \lim_{t \rightarrow \infty} \frac{1}{-\lambda e^{-\lambda t}}$$

where the last equality follows from L'Hôpital's Rule. Because $e^{-\lambda t}$ tends to infinity as $t \rightarrow \infty$ ($-\lambda > 0$), the fraction tends to 0.

11. The characteristic equation is

$$-\lambda(-p - \lambda) + q = \lambda^2 + p\lambda + q = 0.$$

Solving the quadratic equation, one obtains

$$\lambda = \frac{-p \pm \sqrt{p^2 - 4q}}{2}.$$

(a) Therefore, in order for \mathbf{A} to have two real eigenvalues, p and q must satisfy $p^2 - 4q > 0$.

(b) In order for \mathbf{A} to have complex eigenvalues, p and q must satisfy $p^2 - 4q < 0$.

(c) In order for \mathbf{A} to have only one eigenvalue, p and q must satisfy $p^2 - 4q = 0$.

Therefore, $a + d = 2\lambda_0$, and $ad - bc = \lambda_0^2$.

Now we compute $(\mathbf{A} - \lambda_0\mathbf{I})^2$ using the definition of matrix multiplication. We have

$$\begin{aligned} (\mathbf{A} - \lambda_0\mathbf{I})^2 &= \begin{pmatrix} a - \lambda_0 & b \\ c & d - \lambda_0 \end{pmatrix} \begin{pmatrix} a - \lambda_0 & b \\ c & d - \lambda_0 \end{pmatrix} \\ &= \begin{pmatrix} (a - \lambda_0)^2 + bc & b(a + d - 2\lambda_0) \\ c(a + d - 2\lambda_0) & bc + (d - \lambda_0)^2 \end{pmatrix}. \end{aligned}$$

Since $a + d = 2\lambda_0$, we see that the bottom-left and top-right entries are zero.

Now consider the top-left entry $(a - \lambda_0)^2 + bc$. We have

$$\begin{aligned} (a - \lambda_0)^2 + bc &= a^2 - 2a\lambda_0 + \lambda_0^2 + bc \\ &= a^2 - 2a\lambda_0 + ad - bc + bc, \end{aligned}$$

because $ad - bc = \lambda_0^2$. The right-hand side simplifies to

$$a^2 - 2a\lambda_0 + ad = a(a - 2\lambda_0 + d) = 0$$

because $a + d = 2\lambda_0$.

A similar argument is used to show that the bottom-right entry is zero.

- (b) If \mathbf{V}_0 is an eigenvector, then $\mathbf{V}_1 = (\mathbf{A} - \lambda_0\mathbf{I})\mathbf{V}_0$ is the zero vector. If not, we use the result of part (a) to compute

$$(\mathbf{A} - \lambda_0\mathbf{I})\mathbf{V}_1 = (\mathbf{A} - \lambda_0\mathbf{I})^2\mathbf{V}_0 = \mathbf{0} \text{ (the zero vector).}$$

Consequently, \mathbf{V}_1 is an eigenvector.

17. (a) The characteristic polynomial is

$$(-\lambda)(-1 - \lambda) + 0 = \lambda^2 + \lambda,$$

so the eigenvalues are $\lambda = 0$ and $\lambda = -1$.

- (b) To find the eigenvectors \mathbf{V}_1 associated to the eigenvalue $\lambda = 0$, we must solve $\mathbf{A}\mathbf{V}_1 = 0\mathbf{V}_1 = \mathbf{0}$ where \mathbf{A} is the matrix that defines this linear system. (Note that this is the same calculation we do if we want to locate the equilibrium points.) We get

$$\begin{cases} 2y_1 = 0 \\ -y_1 = 0, \end{cases}$$

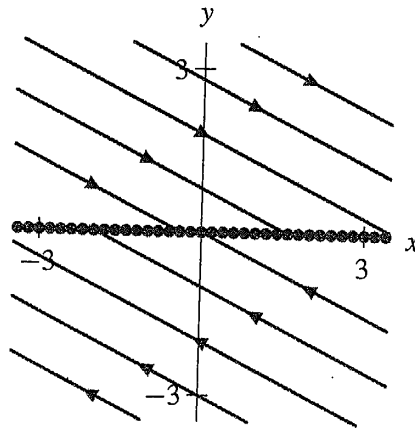
where $\mathbf{V}_1 = (x_1, y_1)$. Hence, the eigenvectors associated to $\lambda = 0$ (as well as the equilibrium points) must satisfy the equation $y_1 = 0$.

To find the eigenvectors \mathbf{V}_2 associated to the eigenvalue $\lambda = -1$, we must solve $\mathbf{A}\mathbf{V}_2 = -\mathbf{V}_2$. We get

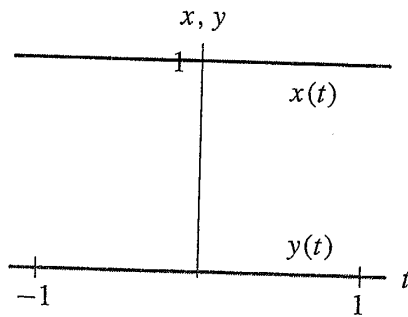
$$\begin{cases} 2y_2 = -x_2 \\ -y_2 = -y_2. \end{cases}$$

where $\mathbf{V}_2 = (x_2, y_2)$. Hence, the eigenvectors associated to $\lambda = -1$ must satisfy $2y_2 = -x_2$.

(c) The equation $y_1 = 0$ specifies a line of equilibrium points. Since the other eigenvalue is negative, solution curves not corresponding to equilibria move toward this line as t increases.



(d) Since $(1, 0)$ is an equilibrium point, it is easy to sketch the corresponding $x(t)$ - and $y(t)$ -graphs.



(e) To form the general solution, we must pick one eigenvector for each eigenvalue. Using part (b), we pick $\mathbf{V}_1 = (1, 0)$, and $\mathbf{V}_2 = (2, -1)$. We obtain the general solution

$$\mathbf{Y}(t) = k_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + k_2 e^{-t} \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

(f) To determine the solution whose initial condition is $(1, 0)$, we can substitute $t = 0$ in the general solution and solve for k_1 and k_2 . However, since this initial condition is an equilibrium point, we need not make the effort. We simply observe that

$$\mathbf{Y}(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

is the desired solution.

18. (a) The characteristic equation is

$$(2 - \lambda)(6 - \lambda) - 12 = \lambda^2 - 8\lambda = 0.$$

Therefore, the eigenvalues are $\lambda = 0$ and $\lambda = 8$.

- (b) To find the eigenvectors \mathbf{V}_1 associated to the eigenvalue $\lambda = 0$, we must solve $\mathbf{A}\mathbf{V}_1 = 0\mathbf{V}_1 = 0$ where \mathbf{A} is the matrix that defines this linear system. (Note that this is the same calculation we do if we want to locate the equilibrium points.) We get

$$\begin{cases} 2x_1 + 4y_1 = 0 \\ 3x_1 + 6y_1 = 0, \end{cases}$$

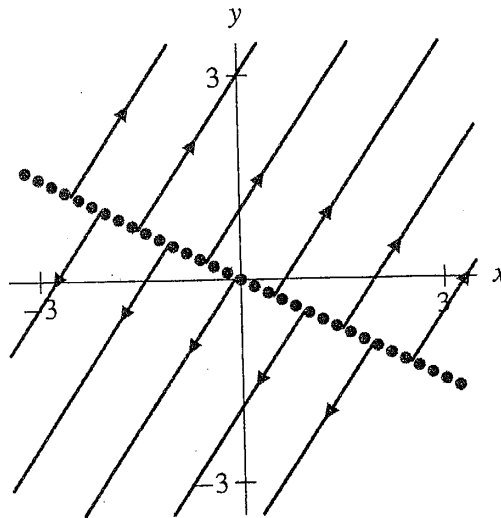
where $\mathbf{V}_1 = (x_1, y_1)$. Hence, the eigenvectors associated to $\lambda = 0$ (as well as the equilibrium points) must satisfy the equation $x_1 + 2y_1 = 0$.

To find the eigenvectors \mathbf{V}_2 associated to the eigenvalue $\lambda = 8$, we must solve $\mathbf{A}\mathbf{V}_2 = 8\mathbf{V}_2$. We get

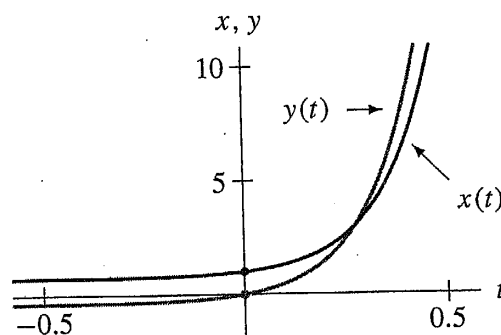
$$\begin{cases} 2x_2 + 4y_2 = 8x_2 \\ 3x_2 + 6y_2 = 8y_2, \end{cases}$$

where $\mathbf{V}_2 = (x_2, y_2)$. Hence, the eigenvectors associated to $\lambda = 8$ must satisfy $2y_2 = 3x_2$.

- (c) The equation $x_1 + 2y_1 = 0$ specifies a line of equilibrium points. Since the other eigenvalue is positive, solution curves not corresponding to equilibria move away from this line as t increases.



- (d) As t increases, both $x(t)$ and $y(t)$ increase exponentially. As t decreases, both x and y approach constants that are determined by the line of equilibrium points.



- (e) To form the general solution, we must pick one eigenvector for each eigenvalue. Using part (b), we pick $\mathbf{V}_1 = (-2, 1)$, and $\mathbf{V}_2 = (2, 3)$. We obtain the general solution

$$\mathbf{Y}(t) = k_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} + k_2 e^{8t} \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

- (f) To determine the solution whose initial condition is $(1, 0)$, we let $t = 0$ in the general solution and obtain the equations

$$k_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} + k_2 \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Therefore, $k_1 = -3/8$ and $k_2 = 1/8$. The particular solution is

$$\mathbf{Y}(t) = \begin{pmatrix} \frac{3}{4} + \frac{1}{4}e^{8t} \\ -\frac{3}{8} + \frac{3}{8}e^{8t} \end{pmatrix}.$$

19. (a) The characteristic polynomial is

$$(4 - \lambda)(1 - \lambda) - 4 = \lambda^2 - 5\lambda,$$

so the eigenvalues are $\lambda = 0$ and $\lambda = 5$.

- (b) To find the eigenvectors \mathbf{V}_1 associated to the eigenvalue $\lambda = 0$, we must solve $\mathbf{A}\mathbf{V}_1 = 0\mathbf{V}_1 = 0$ where \mathbf{A} is the matrix that defines this linear system. (Note that this is the same calculation we do if we want to locate the equilibrium points.) We get

$$\begin{cases} 4x_1 + 2y_1 = 0 \\ 2x_1 + y_1 = 0, \end{cases}$$

where $\mathbf{V}_1 = (x_1, y_1)$. Hence, the eigenvectors associated to $\lambda = 0$ (as well as the equilibrium points) must satisfy the equation $y_1 = -2x_1$.

To find the eigenvectors \mathbf{V}_2 associated to the eigenvalue $\lambda = 5$, we must solve $\mathbf{A}\mathbf{V}_2 = 5\mathbf{V}_2$. We get

$$\begin{cases} 4x_2 + 2y_2 = 5x_2 \\ 2x_2 + y_2 = 5y_2. \end{cases}$$

where $\mathbf{V}_2 = (x_2, y_2)$. Hence, the eigenvectors associated to $\lambda = 5$ must satisfy $x_2 = 2y_2$.

- (c) The equation $y_1 = -2x_1$ specifies a line of equilibrium points. Since the other eigenvalue is positive, solution curves not corresponding to equilibria move away from this line as t increases.