EXERCISES FOR SECTION 3.1

1. Since $a > 0$, Paul's making a profit ($x > 0$) has a beneficial effect on Paul's profits in the future because the ax term makes a positive contribution to dx/dt. However, since $b < 0$, Bob's making a profit ($y > 0$) hinders Paul's ability to make profit because the by term contributes negatively to dx/dt. Roughly speaking, business is good for Paul if his store is profitable and Bob's is not. In fact, since $dx/dt = x - y$, Paul's profits will increase whenever his store is more profitable than Bob's.

Even though $dx/dt = dy/dt = x - y$ for this choice of parameters, the interpretation of the equation is exactly the opposite from Bob's point of view. Since $d < 0$, Bob's future profits are hurt whenever he is profitable because $dy < 0$. But Bob's profits are helped whenever Paul is profitable since $cx > 0$. Once again, since $dy/dt = x - y$, Bob's profits will increase whenever Paul's store is more profitable than his.

Finally, note that both x and y change by identical amounts since dx/dt and dy/dt are always equal.

2. Since $a = 2$, Paul's making a profit ($x > 0$) has a beneficial effect on Paul's future profits because the ax term makes a positive contribution to dx/dt. However, since $b = -1$, Bob's making a profit ($y > 0$) hinders Paul's ability to make profit because the by term contributes negatively to dx/dt. In some sense, Paul's profitability has twice the impact on his profits as does Bob's profitability. For example, Paul's profits will increase whenever his profits are at least one-half of Bob's profits since $dx/dt = 2x - y$.

Since $c = d = 0$, $dy/dt = 0$. Consequently, Bob's profits are not affected by the profitability of either store, and hence his profits are constant in this model.

3. Since $a = 1$ and $b = 0$, we have $dx/dt = x$. Hence, if Paul is making a profit ($x > 0$), then those profits will increase since dx/dt is positive. However, Bob's profits have no effect on Paul's profits. (Note that $dx/dt = x$ is the standard exponential growth model.)

Since $c = 2$ and $d = 1$, profits from both stores have a positive effect on Bob's profits. In some sense, Paul's profits have twice the impact of Bob's profits on dy/dt.

4. Since $a = -1$ and $b = 2$, Paul's making a profit has a negative effect on his future profits. However, if Bob makes a profit, then Paul's profits benefit. Moreover, Bob's profitability has twice the impact as does Paul's. In fact, since $dx/dt = -x + 2y$, Paul's profits will increase if $-x + 2y > 0$ or, in other words, if Bob's profits are at least one-half of Paul's profits.

Since $c = 2$ and $d = -1$, Bob is in the same situation as Paul. His profits contribute negatively to dy/dt since $d = -1$. However, Paul's profitability has twice the positive effect.

Note that this model is symmetric in the sense that both Paul and Bob perceive each others profits in the same way. This symmetry comes from the fact that $a = d$ and $b = c$.

5. $\mathbf{Y} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \dfrac{d\mathbf{Y}}{dt} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \mathbf{Y}$

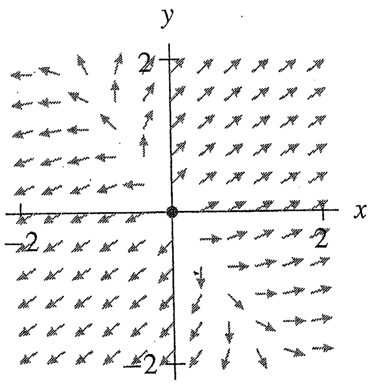
6. $\mathbf{Y} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \dfrac{d\mathbf{Y}}{dt} = \begin{pmatrix} 0 & 3 \\ -0.3 & 3\pi \end{pmatrix} \mathbf{Y}$

7. $\mathbf{Y} = \begin{pmatrix} p \\ q \\ r \end{pmatrix}, \quad \dfrac{d\mathbf{Y}}{dt} = \begin{pmatrix} 3 & -2 & -7 \\ -2 & 0 & 6 \\ 0 & 7.3 & 2 \end{pmatrix} \mathbf{Y}$

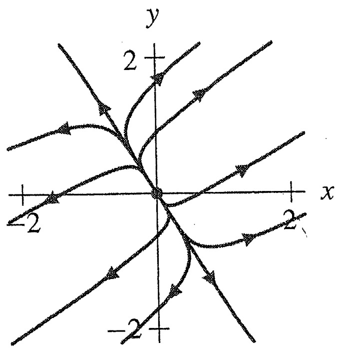
8. $$\frac{dx}{dt} = -3x + 2\pi y$$
$$\frac{dy}{dt} = 4x - y$$

9. $$\frac{dx}{dt} = \beta y$$
$$\frac{dy}{dt} = \gamma x - y$$

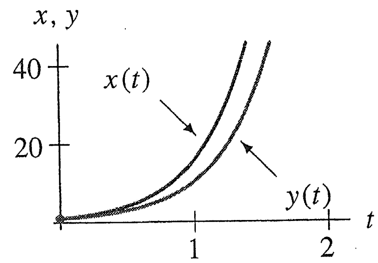
10. (a)



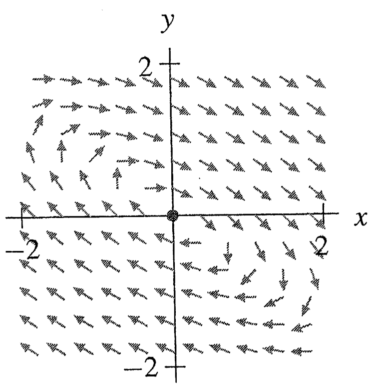
(b)



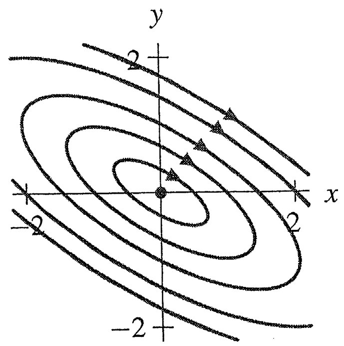
(c)



11. (a)



(b)



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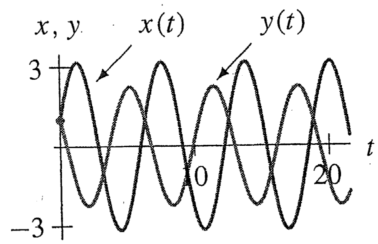
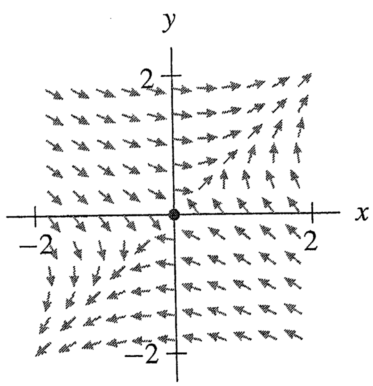
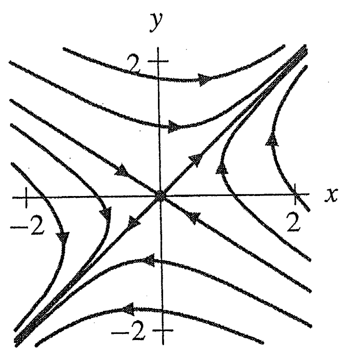


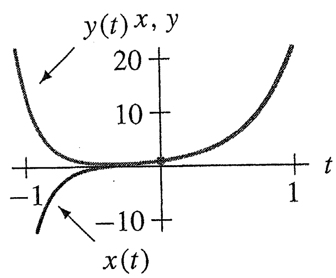
12. (a)



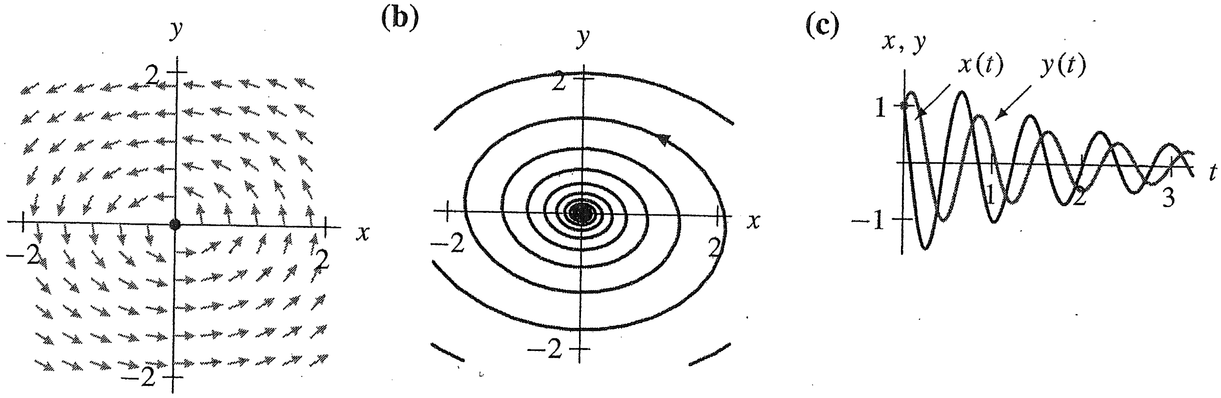
(b)



(c)



13. **(a)** **(b)** **(c)**



14. **(a)** If $a = 0$, then $\det \mathbf{A} = ad - bc = bc$. Thus both b and c are nonzero if $\det \mathbf{A} \neq 0$.

(b) Equilibrium points (x_0, y_0) are solutions of the simultaneous system of linear equations

$$\begin{cases} ax_0 + by_0 = 0 \\ cx_0 + dy_0 = 0. \end{cases}$$

If $a = 0$, the first equation reduces to $by_0 = 0$, and since $b \neq 0$, $y_0 = 0$. In this case, the second equation reduces to $cx_0 = 0$, so $x_0 = 0$ as well. Therefore, $(x_0, y_0) = (0, 0)$ is the only equilibrium point for the system.

15. The vector field at a point (x_0, y_0) is (ax_0+by_0, cx_0+dy_0), so in order for a point to be an equilibrium point, it must be a solution to the system of simultaneous linear equations

$$\begin{cases} ax_0 + by_0 = 0 \\ cx_0 + dy_0 = 0. \end{cases}$$

If $a \neq 0$, we know that the first equation is satisfied if and only if

$$x_0 = -\frac{b}{a} y_0.$$

Now we see that any point that lies on this line $x_0 = (-b/a)y_0$ also satisfies the second linear equation $cx_0 + dy_0 = 0$. In fact, if we substitute a point of this form into the second component of the vector field, we have

$$\begin{aligned} cx_0 + dy_0 &= c\left(-\frac{b}{a}\right) y_0 + dy_0 \\ &= \left(-\frac{bc}{a} + d\right) y_0 \\ &= \left(\frac{ad - bc}{a}\right) y_0 \\ &= \frac{\det \mathbf{A}}{a} y_0 \\ &= 0, \end{aligned}$$

since we are assuming that $\det \mathbf{A} = 0$. Hence, the line $x_0 = (-b/a)y_0$ consists entirely of equilibrium points.

If $a = 0$ and $b \neq 0$, then the determinant condition $\det \mathbf{A} = ad - bc = 0$ implies that $c = 0$. Consequently, the vector field at the point (x_0, y_0) is (by_0, dy_0). Since $b \neq 0$, we see that we get equilibrium points if and only if $y_0 = 0$. In other words, the set of equilibrium points is exactly the x-axis.

Finally, if $a = b = 0$, then the vector field at the point (x_0, y_0) is $(0, cx_0 + dy_0)$. In this case, we see that a point (x_0, y_0) is an equilibrium point if and only if $cx_0 + dy_0 = 0$. Since at least one of c or d is nonzero, the set of points (x_0, y_0) that satisfy $cx_0 + dy_0 = 0$ is precisely a line through the origin.

16. **(a)** Let $v = dy/dt$. Then $dv/dt = d^2y/dt^2 = -qy - p(dy/dt) = -qy - pv$. Thus we obtain the system

$$\begin{aligned} \frac{dy}{dt} &= v \\ \frac{dv}{dt} &= -qy - pv. \end{aligned}$$

In matrix form, this system is written as

$$\begin{pmatrix} \dfrac{dy}{dt} \\ \dfrac{dv}{dt} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix} \begin{pmatrix} y \\ v \end{pmatrix}.$$

(b) The determinant of this matrix is q. Hence, if $q \neq 0$, we know that the only equilibrium point is the origin.

(c) If y is constant, then $v = dy/dt$ is identically zero. Hence, $dv/dt = 0$.

Also, the system reduces to

$$\begin{pmatrix} \dfrac{dy}{dt} \\ \dfrac{dv}{dt} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix} \begin{pmatrix} y \\ 0 \end{pmatrix},$$

which implies that $dv/dt = -qy$.

Combining these two observations, we obtain $dv/dt = -qy = 0$, and if $q \neq 0$, then $y = 0$.

17. The first-order system corresponding to this equation is

$$\begin{aligned} \frac{dy}{dt} &= v \\ \frac{dv}{dt} &= -qy - pv. \end{aligned}$$

(a) If $q = 0$, then the system becomes

$$\begin{aligned} \frac{dy}{dt} &= v \\ \frac{dv}{dt} &= -pv, \end{aligned}$$