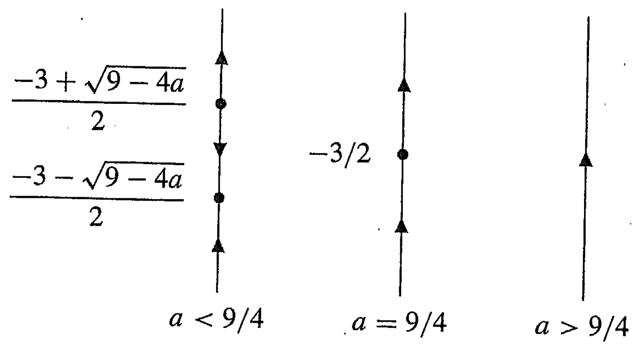
2. The equilibrium points occur at solutions of $dy/dt = y^2 + 3y + a = 0$. From the quadratic formula, we have

$$y = \frac{-3 \pm \sqrt{9 - 4a}}{2}.$$

Hence, the bifurcation value of a is $9/4$. For $a < 9/4$, there are two equilibria, one source and one sink. For $a = 9/4$, there is one equilibrium which is a node, and for $a > 9/4$, there are no equilibria.



Phase lines for $a < 9/4$, $a = 9/4$, and $a > 9/4$.

3. The equilibrium points occur at solutions of $dy/dt = y^2 - ay + 1 = 0$. From the quadratic formula, we have

$$y = \frac{a \pm \sqrt{a^2 - 4}}{2}.$$

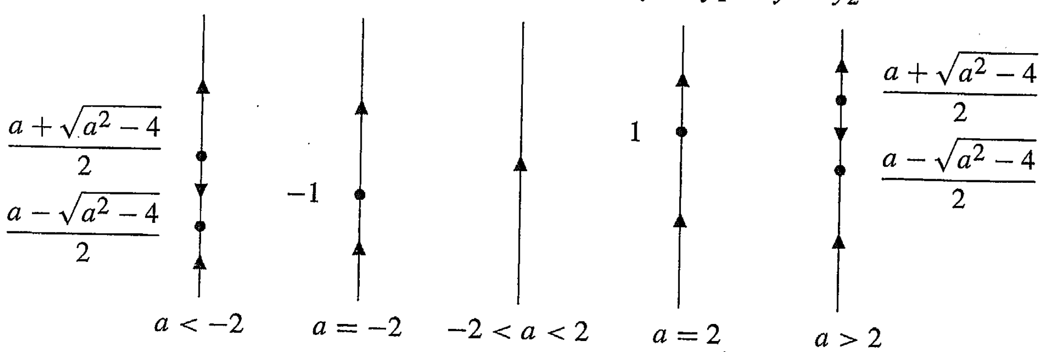
If $-2 < a < 2$, then $a^2 - 4 < 0$, and there are no equilibrium points. If $a > 2$ or $a < -2$, there are two equilibrium points. For $a = \pm 2$, there is one equilibrium point at $y = a/2$. The bifurcations occur at $a = \pm 2$.

To draw the phase lines, note that:

- For $-2 < a < 2$, $dy/dt = y^2 - ay + 1 > 0$, so the solutions are always increasing.
- For $a = 2$, $dy/dt = (y - 1)^2 \geq 0$, and $y = 1$ is a node.
- For $a = -2$, $dy/dt = (y + 1)^2 \geq 0$, and $y = -1$ is a node.
- For $a < -2$ or $a > 2$, let

$$y_1 = \frac{a - \sqrt{a^2 - 4}}{2} \quad \text{and} \quad y_2 = \frac{a + \sqrt{a^2 - 4}}{2}.$$

Then $dy/dt < 0$ if $y_1 < y < y_2$, and $dy/dt > 0$ if $y < y_1$ or $y > y_2$.

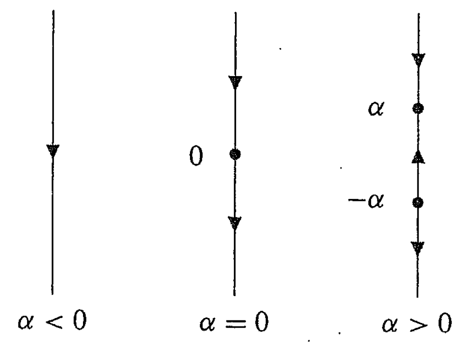


The five possible phase lines.

6. The equilibrium points occur at solutions of $dy/dt = \alpha - |y| = 0$. For $\alpha < 0$, there are no equilibrium points. For $\alpha = 0$, there is one equilibrium point, $y = 0$. For $\alpha > 0$, there are two equilibrium points, $y = \pm\alpha$. Therefore, $\alpha = 0$ is a bifurcation value.

To draw the phase lines, note that:

- If $\alpha < 0$, $dy/dt = \alpha - |y| < 0$, so the solutions are always decreasing.
- If $\alpha = 0$, $dy/dt < 0$ unless $y = 0$. Thus, $y = 0$ is a node.
- For $\alpha > 0$, $dy/dt > 0$ for $-\alpha < y < \alpha$, and $dy/dt < 0$ for $y < -\alpha$ and for $y > \alpha$.



7. The bifurcations occur at values of α for which the graph of $\sin y + \alpha$ is tangent to the y-axis. That is, $\alpha = -1$ and $\alpha = 1$.

For $\alpha < -1$, there are no equilibria, and all solutions become unbounded in the negative direction as t increases.

If $\alpha = -1$, there are equilibrium points at $y = \pi/2 \pm 2n\pi$ for every integer n. All equilibria are nodes, and as $t \to \infty$, all other solutions decrease toward the nearest equilibrium solution below the given initial condition.

For $-1 < \alpha < 1$, there are infinitely many sinks and infinitely many sources, and they alternate along the phase line. Successive sinks differ by 2π. Similarly, successive sources are separated by 2π.

As α increases from -1 to $+1$, nearby sink and source pairs move apart. This separation continues until α is close to 1 where each source is close to the next sink with larger value of y.

At $\alpha = 1$, there are infinitely many nodes, and they are located at $y = 3\pi/2 \pm 2n\pi$ for every integer n. For $\alpha > 1$, there are no equilibria, and all solutions become unbounded in the positive direction as t increases.

8. Note that $0 < e^{-y^2} \leq 1$ for all y, and its maximum value occurs at $y = 0$. Therefore, for $\alpha < -1$, dy/dt is always negative, and the solutions are always decreasing.

If $\alpha = -1$, $dy/dt = 0$ if and only if $y = 0$. For $y \neq 0$, $dy/dt < 0$, and the equilibrium point at $y = 0$ is a node.

If $-1 < \alpha < 0$, then there are two equilibrium points which we compute by solving

$$e^{-y^2} + \alpha = 0.$$

We get $-y^2 = \ln(-\alpha)$. Consequently, $y = \pm\sqrt{\ln(-1/\alpha)}$. As $\alpha \to 0$ from below, $\ln(-1/\alpha) \to \infty$, and the two equilibria tend to $\pm\infty$.

If $\alpha \geq 0$, dy/dt is always positive, and the solutions are always increasing.

9. For $\alpha = 0$, there are three equilibria. There is a sink to the left of $y = 0$, a source at $y = 0$, and a sink to the right of $y = 0$.

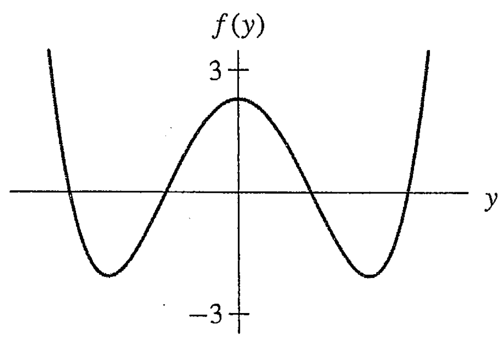
As α decreases, the source and sink on the right move together. A bifurcation occurs at $\alpha \approx -2$. At this bifurcation value, there is a sink to the left of $y = 0$ and a node to the right of $y = 0$. For α below this bifurcation value, there is only the sink to the left of $y = 0$.

As α increases from zero, the sink to the left of $y = 0$ and the source move together. There is a bifurcation at $\alpha \approx 2$ with a node to the left of $y = 0$ and a sink to the right of $y = 0$. For α above this bifurcation value, there is only the sink to the right of $y = 0$.

10. Note that if α is very negative, then the equation $g(y) = -\alpha y$ has only one solution. It is $y = 0$. Furthermore, $dy/dt > 0$ for $y < 0$, and $dy/dt < 0$ for $y > 0$. Consequently, the equilibrium point at $y = 0$ is a sink.

In the figure, it appears that the tangent line to the graph of g at the origin has slope 1 and does not intersect the graph of g other than at the origin. If so, $\alpha = -1$ is a bifurcation value. For $\alpha \leq -1$, the differential equation has one equilibrium, which is a sink. For $\alpha > -1$, the equation has three equilibria, $y = 0$ and two others, one on each side of $y = 0$. The equilibrium point at the origin is a source, and the other two equilibria are sinks.

11. The graph of f needs to cross the y-axis exactly four times so that there are exactly four equilibria if $\alpha = 0$. The function must be greater than -3 everywhere so that there are no equilibria if $\alpha \geq 3$. Finally, the graph of f must cross horizontal lines three or more units above the y-axis exactly twice so that there are exactly two equilibria for $\alpha \leq -3$. The following graph is an example of the graph of such a function.

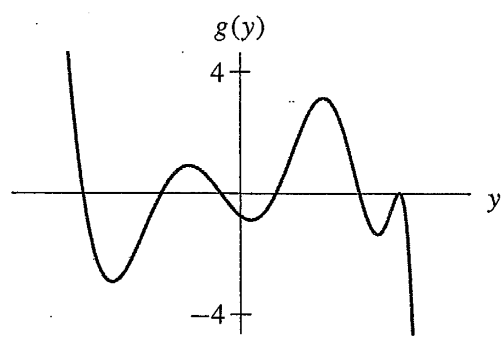


12. The graph of g can only intersect horizontal lines above 4 once, and it must go from above to below as y increases. Then there is exactly one sink for $\alpha \leq -4$.

Similarly, the graph of g can only intersect horizontal lines below -4 once, and it must go from above to below as y increases. Then there is exactly one sink for $\alpha \geq 4$.

Finally, the graph of g must touch the y-axis at exactly six points so that there are exactly six equilibria for $\alpha = 0$.

The following graph is the graph of one such function.



13. No such $f(y)$ exists. To see why, suppose that there is exactly one sink y_0 for $\alpha = 0$. Then, $f(y) > 0$ for $y < y_0$, and $f(y) < 0$ for $y > y_0$. Now consider the system $dy/dt = f(y) + 1$. Then $dy/dt \geq 1$ for $y < y_0$. If this system has an equilibrium point y_1 that is a source, then $y_1 > y_0$ and $dy/dt < 0$ for y slightly less than y_1. Since $f(y)$ is continuous and $dy/dt \geq 1$ for $y \leq y_0$, then dy/dt must have another zero between y_0 and y_1.

14. No, it is not possible to find such a function $g(y)$.

To see why, consider the two cases $\alpha = -1$ and $\alpha = +1$. If $\alpha = -1$ and the equation has exactly one sink y_0 and no other equilibrium points, then $g(y) > 1$ for $y < y_0$. If $\alpha = 1$ and the equation has exactly three equilibria—one sink and two sources, then the sink must be between the two sources. Let y_1 be the lower source. Since $dy/dt = g(y) + 1$, we see that $dy/dt > 0$ for all y less than the minimum of y_0 and y_1. Therefore, y_1 cannot be a source with no other equilibrium points below it.

15. **(a)** For all $C \geq 0$, the equation has a source at $P = C/k$, and this is the only equilibrium point. Hence all of the phase lines are qualitatively the same, and there are no bifurcation values for C.

(b) If $P(0) > C/k$, the corresponding solution $P(t) \to \infty$ at an exponential rate as $t \to \infty$, and if $P(0) < C/k$, $P(t) \to -\infty$, passing through "extinction" ($P = 0$) after a finite time.

16. First we set $C = 0$ so that the population can grow. Once the population reaches the desired level ($P = 100{,}000$), then we set $C = 100{,}000k = 200{,}000$. With this value of C, the population $P = 100{,}000$ is an equilibrium point.

We must be diligent in our management of the population since the equilibrium point is a source. A small change in P due to random fluctuations will eventually cause extinction or explosion of the population.

17. **(a)** A model of the fish population that includes fishing is

$$\frac{dP}{dt} = 2P - \frac{P^2}{50} - 3L,$$

where L is the number of licenses issued. The coefficient of 3 represents the average catch of 3 fish per year. As L is increased, the two equilibrium points for $L = 0$ (at $P = 0$ and $P = 100$) will move together. If L is sufficiently large, there are no equilibrium points. Hence we wish to pick L as large as possible so that there is still an equilibrium point present. In other words, we want the bifurcation value of L. The bifurcation value of L occurs if the equation

$$\frac{dP}{dt} = 2P - \frac{P^2}{50} - 3L = 0$$

has just one solution for P in terms of L. Using the quadratic formula, we see that there is exactly one equilibrium point if $L = 50/3$. Since this value of L is not an integer, the largest number of licenses that should be allowed is 16.

(b) If we allow the fish population to come to equilibrium then the population will be at the carrying capacity, which is $P = 100$ if $L = 0$. If we then allow 16 licenses to be issued, we expect that the population is a solution to the new model with $L = 16$ and initial population $P = 100$. The model becomes

$$\frac{dP}{dt} = 2P - \frac{P^2}{50} - 48,$$

which has a source at $P = 40$ and a sink at $P = 60$.