Test 1: Differential Equations

Math 341 Fall 2008

Friday October 3
2:30pm-3:25pm

Name: BUCKMIRE

Directions: Read *all* problems first before answering any ~~of~~ them. There are 6 pages in this test. This is a 55-minute, no-notes, closed book, test. **No calculators.** You must show all relevant work to support your answers. Use complete English sentences as much as possible and CLEARLY indicate your final answers to be graded from your "scratch work."

Pledge: I, ______________________________, pledge my honor as a human being and Occidental student, that I have followed all the rules above to the letter and in spirit.

No.	Score	Maximum
1		24
2		16
3		30
4		30
Total		**100**

1. *[24 points total.]* **Existence and Uniqueness.**

(a) *[12 points].* Write down an example of an ordinary differential equation $y' = f(t, y)$ AND an initial condition $y(t_0) = y_0$. This initial value problem you come up with should **violate** the hypothesis of the Existence and Uniqueness Theorem that guarantees uniqueness at (t_0, y_0). Solve your initial value problem and explain what (if anything) the Existence and Uniqueness Theorem allows you to conclude about the solution of your IVP.

Uniqueness hypothesis is that $f_y(t,y)$ is continuous at (t_0,y_0).

We can conclude NOTHING from the theorem since its hypothesis is not satisfied!

Choose

$y' = f(y) = \sqrt{y}$

$y(0) = 0$

$f'(t,y) = \frac{1}{2\sqrt{y}}$ is NOT continuous at $(1,0)$ (divide by zero)

$\frac{dy}{dt} = \sqrt{y} \Rightarrow \int \frac{dy}{\sqrt{y}} = \int dt \Rightarrow 2\sqrt{y} = t + C \Rightarrow \sqrt{y} = \frac{t}{2} \Rightarrow y = \frac{t^2}{4}$

$0 = 0 + C$

$0 = C$

$y = 0$ is also a solution

(b) *[12 points].* Write down an example of an ordinary differential equation $y' = f(t, y)$ AND an initial condition $y(t_0) = y_0$. This initial value problem you come up with should **satisfy** the hypothesis of the Existence and Uniqueness Theorem that guarantees uniqueness at (t_0, y_0). Solve your initial value problem and explain what (if anything) the Existence and Uniqueness Theorem allows you to conclude about your solution of your IVP.

$y' = y$

$y(0) = 1$

$f(t,y) = y$

$\frac{\partial f}{\partial y} = 1$ is continuous everywhere since its a constant

$y(t) = e^t$

is solution.

Since the hypothesis IS satisfied I know this is the ONLY solution to $\{y' = y, y(0) = 1\}$

2. *[16 points total.]* **Mathematical Modeling, Euler's Method**
Suppose the population P of Tribbles in the universe is modelled by the relationship that their growth rate $\frac{dP}{dt}$ is directly proportional to the square of the current number of Tribbles. Initially there are exactly two Tribbles. On Friday (today) you discover that when there are only 10 known Tribbles in the universe their growth rate is 1 Tribble per day.

(a) *[6 points].* Write down an initial value problem that represents the mathematical model for the population of Tribbles in the universe at any time.

$$P' \propto P^2 \Rightarrow P' = kP^2, \; P(0) = 2$$

On Friday $P' = 1 = k \cdot 10^2 \Rightarrow k = \frac{1}{100}$

$$\boxed{P' = \frac{1}{100} P^2, \; P(0) = 2}$$

(b) *[5 points].* Use Euler's Method to estimate the number of Tribbles exactly two days from now (on Sunday).

$$P(\text{Sunday}) \approx P(\text{Friday}) + P'(\text{Friday})\, 2\text{ days}$$
$$\approx 10 + 1 \cdot 2$$
$$\approx 12$$

OR

$$P(2) \approx P(0) + P'(0) \cdot 2$$
$$\approx 2 + \left(\frac{1}{100} 2^2\right) \cdot 2 \approx 2.08 \text{ tribbles}$$

(c) *[5 points].* Is your Euler's Method approximation from (b) over-estimating or under-estimating the number of Tribbles in the universe? EXPLAIN YOUR ANSWER.

You could either (1) Solve the system exactly and compare your approx to the exact,
or (2) Calculate the concavity of P

(1) $\frac{dP}{P^2} = \frac{1}{100} dt$

$-\frac{1}{P} = \frac{t}{100} + c$

$P(0) = 2$

$-\frac{1}{2} = c$

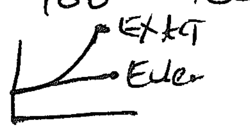
$-\frac{1}{P} = \frac{t}{100} - \frac{1}{2} \Rightarrow \frac{t-50}{100} = -\frac{1}{P}$

$P = \frac{100}{50-t}$

$P(2) = \frac{100}{48} = 2\frac{4}{48} = 2\frac{1}{12}$

(2) $P'' = \frac{1}{100} 2PP' = \frac{1}{100} 2P \cdot \frac{P^2}{100}$

$P''(\text{Sunday}) > 0$



Thus Euler's Method will be UNDER ESTIMATE

3. *[30 points total.]* **Systems of ODEs, Analytical Techniques, Linearity Principle.** Are the following statements TRUE or FALSE – put your answer in the box. To receive ANY credit, you must also give a brief, and correct, explanation in support of your answer! For example, if you think the answer is **FALSE** providing a counterexample for which the statement is NOT TRUE is best. If you think the answer is **TRUE** you should prove why you think the statement is always true. Your explanation of your answer is worth FOUR TIMES as much as the answer you put in the box.

(a) **TRUE or FALSE?** The initial value problem $y' + 2y = 3, y(0) = 1$ possesses the one-parameter family of solutions $y(x) = \frac{1}{2}(3 - Ce^{-2x})$.

$\boxed{\text{FALSE}}$

IVPs have one solution (according to Existence & Uniqueness)

If there was no initial condition, then...

$Ly = \left(\frac{d}{dt} + 2\right)y = 3$

$Ly_h = 0 \Rightarrow y_h = Ce^{-2x}$

$Ly_p = 3$

$y_p = A + Bt$

$Ly_p = B + 2A + 2Bt = 3$

$\Rightarrow B = 0$

$A = 3/2 \Rightarrow y_p = \frac{3}{2}$

$y_g = \frac{3}{2} + Ce^{-2x}$

$\Rightarrow y_g = \frac{1}{2}(3 - Ae^{-2x})$

(b) **TRUE or FALSE?** For the system $x' = -3x$, $y' = -2y$ a solution starting at (a, b) when $t = 0$ will approach the origin as $t \to \infty$ for all non-zero real valuues of a and b.

$\boxed{\text{TRUE}}$

$x' = -3x,\ x(0) = a \Rightarrow x(t) = ae^{-3t}$

$y' = -2y,\ y(0) = b \Rightarrow y(t) = be^{-2t}$

$$\lim_{t \to \infty} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \lim_{t \to \infty} \begin{pmatrix} ae^{-3t} \\ be^{-2t} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

(c) **TRUE or FALSE?** The linear ordinary differential equation $\frac{dy}{dt} = 2 - y$ has solutions of the form $y_1(t) + y_2(t)$ where $y_1(t) = 2$ and $y_2(t)$ is *any* real number multiple of e^{-t}.

$\boxed{\text{TRUE}}$

The homogeneous solution is ce^{-t}

$Ly = \frac{dy}{dt} + y = 2$

$Ly_h = 0 \Rightarrow y_h = ce^{-t}$

$Ly_p = 2$

$y_p = 2$ is a sol^n

$y_g = y_h + y_p = ce^{-t} + 2$

4. *[30 pts. total]* **Phase Lines, Equilibria, Bifurcations.**

Consider the following differential equation: $\dfrac{dy}{dt} = y^3 + \alpha y^2$, where α is a parameter.

(a) *[6 pts]* What are the equilibrium values of y? Identify them as y^*.

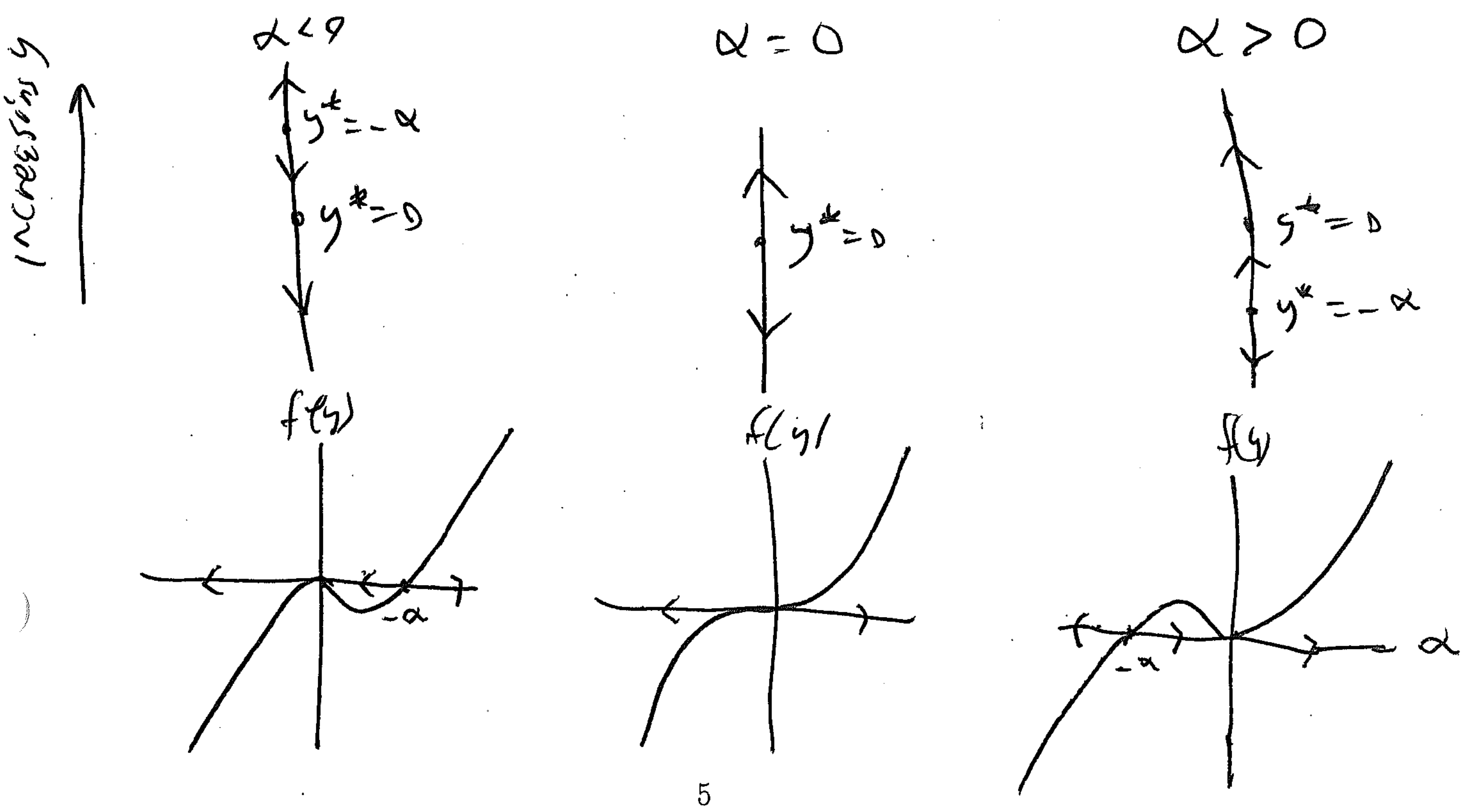
$$f(y;\alpha) = y^3 + \alpha y^2 = 0$$
$$= y^2(y+\alpha) = 0$$
$$y = 0 \text{ or } -\alpha$$
$$y^* = 0,\ y^* = -\alpha$$

(b) *[10 points]* Is there a bifurcation value for the parameter α? If so, identify it as α_B and draw phase lines corresponding to the case where $\alpha < \alpha_B$, $\alpha = \alpha_B$ and $\alpha > \alpha_B$. Indicate locations of y^* on your phase lines.

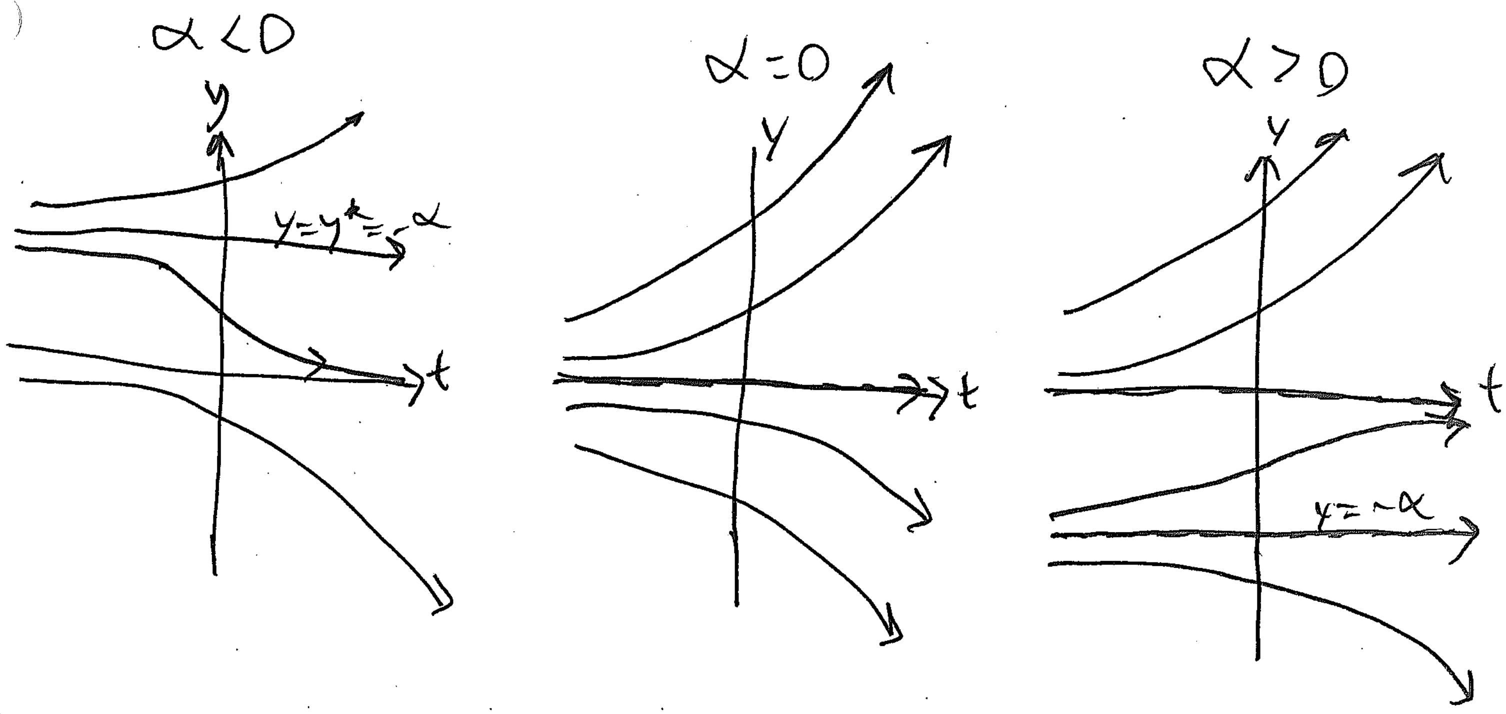
bifurcation occurs when $f(y;\alpha) = 0$ AND $f'(y;\alpha) = 0$

$$f'(y;\alpha) = 3y^2 + 2\alpha y = 0$$
$$(3y + 2\alpha)y = 0 \quad y = 0 \text{ or } y = -\frac{2\alpha}{3}$$
$$-\frac{2\alpha}{3} = -\alpha \Rightarrow \alpha = 0$$

$\alpha = 0$ is α_B



(c) *[6 points]* Sketch representative solutions in the ty-plane corresponding to each of the phase lines you drew in part (b).



(d) *[8 points]* Draw a bifurcation diagram for the differential equation in the αy-plane. Indicate clearly where sinks, sources and nodes occur.