Test 1: Differential Equations

Math 341 Fall 2008
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Friday October 3
2:30pm-3:25pm

Directions: Read all problems first before answering any of them. There are 6 pages in this test. This is a 55-minute, no-notes, closed book, test. No calculators. You must show all relevant work to support your answers. Use complete English sentences as much as possible and CLEARLY indicate your final answers to be graded from your “scratch work.”

Pledge: I, __________________________, pledge my honor as a human being and Occidental student, that I have followed all the rules above to the letter and in spirit.

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1. [24 points total.] Existence and Uniqueness.
(a) [12 points]. Write down an example of an ordinary differential equation \( y' = f(t, y) \) AND an initial condition \( y(t_0) = y_0 \). This initial value problem you come up with should violate the hypothesis of the Existence and Uniqueness Theorem that guarantees uniqueness at \((t_0, y_0)\). Solve your initial value problem and explain what (if anything) the Existence and Uniqueness Theorem allows you to conclude about the solution of your IVP.

Uniqueness hypothesis is that \( f_y(y, t) \) is continuous at \((t_0, y_0)\).

Choose
\[
\begin{align*}
y' &= f(y) = \sqrt{y} \\
y(0) &= 0
\end{align*}
\]

is not continuous at \((1, 0)\) (divide by zero)

\[
\frac{dy}{dt} = \frac{dy}{\sqrt{y}} \Rightarrow \int \frac{dy}{\sqrt{y}} = \int dt = \frac{2}{2} \Rightarrow y = \sqrt{t} + C
\]

\( y = 0 \) is also a solution.

(b) [12 points]. Write down an example of an ordinary differential equation \( y' = f(t, y) \) AND an initial condition \( y(t_0) = y_0 \). This initial value problem you come up with should satisfy the hypothesis of the Existence and Uniqueness Theorem that guarantees uniqueness at \((t_0, y_0)\). Solve your initial value problem and explain what (if anything) the Existence and Uniqueness Theorem allows you to conclude about your solution of your IVP.

\[
\begin{align*}
y' &= y \\
y(0) &= 1
\end{align*}
\]

\( f(t, y) = y \)

\( \frac{dy}{dt} = 1 \) is continuous everywhere since it's a constant.

\( y(t) = e^t \)

is solution.

Since the hypothesis IS satisfied, I know this is the ONLY solution to \( \{ y' = y, y(0) = 1 \} \).
2. **[16 points total]** Mathematical Modeling, Euler's Method
Suppose the population $P$ of Tribbles in the universe is modelled by the relationship that their growth rate $\frac{dP}{dt}$ is directly proportional to the square of the current number of Tribbles. Initially there are exactly two Tribbles. On Friday (today) you discover that when there are only 10 known Tribbles in the universe their growth rate is 1 Tribble per day.

(a) **[6 points]**. Write down an initial value problem that represents the mathematical model for the population of Tribbles in the universe at any time.

\[
P' \propto P^2 \Rightarrow P' = kP^2, \quad P(0) = 2
\]

On Friday \(P' = 1 = k \cdot 10^2 \Rightarrow k = \frac{1}{100}\)

\[
P' = \frac{1}{100} P^2, \quad P(0) = 2
\]

(b) **[5 points]**. Use Euler's Method to estimate the number of Tribbles exactly two days from now (on Sunday).

\[
P(\text{Sunday}) \approx P(\text{Friday}) + P'(\text{Friday}) \cdot 2 \text{ days}
\]

\[
\approx 10 + 1.2
\]

\[
\approx 12
\]

OR

\[
P(2) \approx P(0) + P'(0) \cdot 2
\]

\[
\approx 2 + \left(\frac{1}{100} \cdot 2^2\right) \cdot 2 \approx 2.08 \text{ Tribbles}
\]

(c) **[5 points]**. Is your Euler's Method approximation from (b) over-estimating or under-estimating the number of Tribbles in the universe? EXPLAIN YOUR ANSWER.

You could either (1) Solve the system exactly and compare your approach to the exact, or (2) Calculate the concavity at $P$.

\[\frac{dP}{dt} = \frac{1}{100} P^2\]

\[P = \frac{50 - t}{100}, \quad P(0) = 2\]

(2) $P''' = \frac{1}{100} P^2 \Rightarrow P''(\text{Sunday}) > 0$

Thus Euler's Method will be an underestimate.
3. [30 points total.] Systems of ODEs, Analytical Techniques, Linearity Principle. Are the following statements TRUE or FALSE – put your answer in the box. To receive ANY credit, you must also give a brief, and correct, explanation in support of your answer! For example, if you think the answer is FALSE providing a counterexample for which the statement is NOT TRUE is best. If you think the answer is TRUE you should prove why you think the statement is always true. Your explanation of your answer is worth FOUR TIMES as much as the answer you put in the box.

(a) TRUE or FALSE? The initial value problem \( y' + 2y = 3, y(0) = 1 \) possesses the one-parameter family of solutions \( y(x) = \frac{1}{2}(3 - Ce^{-2x}) \).

**FALSE**

\[ \text{LVPs have one solution (according to Existence & Uniqueness)} \]

If there was no initial condition, then:

\[ Ly = \left( \frac{d}{dx} + 2 \right) y = 3 \]
\[ Ly_p = 2 \]
\[ y_p = \frac{3}{2} + Ce^{-2x} \]
\[ y_e = A + Bt \]
\[ y = y_p + y_e \]
\[ y(0) = 1 \]
\[ C = 1 \]
\[ \frac{4}{3} + B = 1 \]
\[ B = -\frac{1}{3} \]

(b) TRUE or FALSE? For the system \( x' = -3x, y' = -2y \) a solution starting at \((a, b)\) when \( t = 0 \) will approach the origin as \( t \to \infty \) for all non-zero real values of \( a \) and \( b \).

**TRUE**

\[ x' = -3x, x(0) = a \Rightarrow x(t) = ae^{-3t} \]
\[ y' = -2y, y(0) = b \Rightarrow y(t) = be^{-2t} \]
\[ \lim_{t \to \infty} \left( x(t) \right) = \lim_{t \to \infty} \left( ae^{-3t} \right) = (0) \]
\[ \lim_{t \to \infty} \left( y(t) \right) = \lim_{t \to \infty} \left( be^{-2t} \right) = (0) \]

(c) TRUE or FALSE? The linear ordinary differential equation \( \frac{dy}{dt} = 2 - y \) has solutions of the form \( y_1(t) + y_2(t) \) where \( y_1(t) = 2 \) and \( y_2(t) \) is any real number multiple of \( e^{-t} \).

**TRUE**

The homogeneous solution is \( Ce^{-t} \)

\[ Ly = \frac{dy}{dt} + y = 2 \]
\[ Ly_h = 0 \Rightarrow y_h = Ce^{-t} \]
\[ Ly_p = 2 \]
\[ y_p = 2 \]
\[ y = y_h + y_p = Ce^{-t} + 2 \]
Consider the following differential equation: \( \frac{dy}{dt} = y^3 + \alpha y^2 \), where \( \alpha \) is a parameter.

(a) [6 pts] What are the equilibrium values of \( y \)? Identify them as \( y^* \).

\[
f(y, \alpha) = y^3 + \alpha y^2 = 0
\]

\[
f^`(y, \alpha) = 3y^2 + 2\alpha y = 0
\]

\[
y = 0 \text{ or } -\alpha
\]

\[
y^* = 0, y^* = -\alpha
\]

(b) [10 points] Is there a bifurcation value for the parameter \( \alpha \)? If so, identify it as \( \alpha_B \) and draw phase lines corresponding to the case where \( \alpha < \alpha_B \), \( \alpha = \alpha_B \) and \( \alpha > \alpha_B \). Indicate locations of \( y^* \) on your phase lines.

\[
\text{bifurcation occurs when } f(y, \alpha) = 0 \text{ AND } f^`(y, \alpha) = 0
\]

\[
f^`(y, \alpha) = 3y^2 + 2\alpha y = 0
\]

\[
(3y + 2\alpha)y = 0 \quad y = 0 \quad \alpha y = -\frac{2\alpha}{3}
\]

\[
\alpha = 0 \text{ is } \alpha_B
\]

\[
\alpha < 0 \quad y^* = -\alpha
\]

\[
\alpha = 0 \quad y^* = 0
\]

\[
\alpha > 0 \quad y^* = -\alpha
\]
(c) [6 points] Sketch representative solutions in the $ty$-plane corresponding to each of the phase lines you drew in part (b).

\[ \alpha < 0 \]
\[ \alpha = 0 \]
\[ \alpha > 0 \]

(d) [8 points] Draw a bifurcation diagram for the differential equation in the $\alpha y$-plane. Indicate clearly where sinks, sources and nodes occur.