DIFFERENTIAL EQUATIONS AND BOUNDARY VALUE PROBLEMS

Computing and Modeling

Third Edition

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Critical Points of Linear Systems

We can use the eigenvalue-eigenvector method of Section 5.2 to investigate the critical point (0, 0) of a linear system

➤ $$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \tag{9}$$

with constant coefficient matrix $\mathbf{A}$. Recall that the eigenvalues λ_1 and λ_2 of $\mathbf{A}$ are the solutions of the characteristic equation

➤ $$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix} = (a - \lambda)(d - \lambda) - bc = 0.$$

We assume that (0, 0) is an *isolated* critical point of the system in (9), so it follows that the coefficient determinant $ad - bc$ of the system $ax + by = 0$, $cx + dy = 0$ is *nonzero*. This implies that $\lambda = 0$ is *not* a solution of (9), and hence that both eigenvalues of the matrix $\mathbf{A}$ are nonzero.

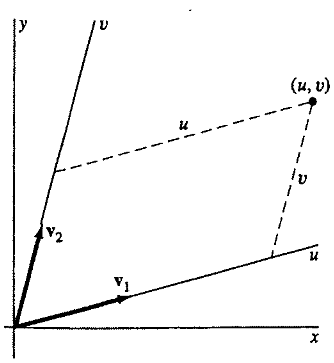
The nature of the isolated critical point (0, 0) then depends on whether the two nonzero eigenvalues λ_1 and λ_2 of $\mathbf{A}$ are

- real and unequal with the same sign;
- real and unequal with opposite signs;
- real and equal;
- complex conjugates with nonzero real part; or
- pure imaginary numbers.

These five cases are discussed separately. In each case the critical point (0, 0) resembles one of those we saw in the examples of Section 6.1—a node (proper or improper), a saddle point, a spiral point, or a center.

UNEQUAL REAL EIGENVALUES WITH THE SAME SIGN: In this case the matrix $\mathbf{A}$ has linearly independent eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ and the general solution $\mathbf{x}(t) = \begin{bmatrix} x(t) & y(t) \end{bmatrix}^T$ of (9) takes the form

$$\mathbf{x}(t) = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t}. \tag{10}$$



IGURE 6.2.3. The oblique v-coordinate system determin- l by the eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$.

This solution is most simply described in the oblique uv-coordinate system indicated in Fig. 6.2.3, in which the u- and v-axes are determined by the eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$. Then the uv-coordinate functions $u(t)$ and $v(t)$ of the moving point $\mathbf{x}(t)$ are simply its distances from the origin measured in the directions parallel to the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$, so it follows from Eq. (10) that a trajectory of the system is described by

$$u(t) = u_0 e^{\lambda_1 t}, \quad v(t) = v_0 e^{\lambda_2 t} \tag{11}$$

where $u_0 = u(0)$ and $v_0 = v(0)$. If $v_0 = 0$, then this trajectory lies on the u-axis, whereas if $u_0 = 0$, then it lies on the v-axis. Otherwise—if u_0 and v_0 are both nonzero—the parametric curve in (11) takes the explicit form $v = Cu^k$ where $k = \lambda_2/\lambda_1 > 0$. These solution curves are tangent at (0, 0) to the u-axis if $k > 1$, to the v-axis if $0 < k < 1$. Thus we have in this case an **improper node** as in Example 3 of Section 6.1. If λ_1 and λ_2 are both positive, then we see from (10) and (11) that these solution curves "depart from the origin" as t increases, so (0, 0) is a **nodal source.** But if λ_1 and λ_2 are both negative, then these solution curves approach the origin as t increases, so (0, 0) is a **nodal sink.**

Example 2 (a) The matrix

$$\mathbf{A} = \frac{1}{8}\begin{bmatrix} 7 & 3 \\ -3 & 17 \end{bmatrix}$$

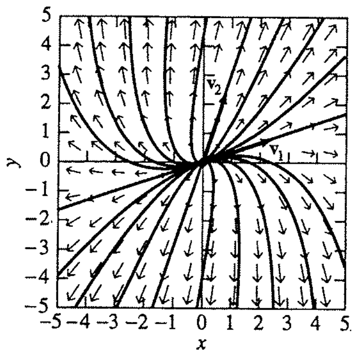


FIGURE 6.2.4. The improper nodal source of Example 2.

has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 2$ with associated eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 3 & 1 \end{bmatrix}^T$ and $\mathbf{v}_2 = \begin{bmatrix} 1 & 3 \end{bmatrix}^T$. Figure 6.2.4 shows a direction field and typical trajectories of the corresponding linear system $\mathbf{x}' = \mathbf{A}\mathbf{x}$. Note that the two eigenvectors point in the directions of the linear trajectories. As is typical of an improper node, all other trajectories are tangent to one of the oblique axes through the origin. In this example the two unequal real eigenvalues are both positive, so the critical point (0, 0) is an improper nodal source.

(b) The matrix

$$\mathbf{B} = -\mathbf{A} = \frac{1}{8}\begin{bmatrix} -7 & -3 \\ 3 & -17 \end{bmatrix}$$

has eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -2$ with the same associated eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 3 & 1 \end{bmatrix}^T$ and $\mathbf{v}_2 = \begin{bmatrix} 1 & 3 \end{bmatrix}^T$. The new linear system $\mathbf{x}' = \mathbf{B}\mathbf{x}$ has the same direction field and trajectories as in Fig. 6.2.4 except with the direction field arrows now all reversed, so (0, 0) is now an improper nodal sink. ■

UNEQUAL REAL EIGENVALUES WITH OPPOSITE SIGNS: Here the situation is the same as in the previous case, except that $\lambda_2 < 0 < \lambda_1$ in (11). The trajectories with $u_0 = 0$ or $v_0 = 0$ lie on the u- and v-axes through the critical point (0, 0). Those with u_0 and v_0 both nonzero are curves of the explicit form $v = Cu^k$, where $k = \lambda_2/\lambda_1 < 0$. As in the case $k < 0$ of Example 3 in Section 6.1, the nonlinear trajectories resemble hyperbolas, and the critical point (0, 0) is therefore an unstable **saddle point.**

Example 3 The matrix

$$\mathbf{A} = \frac{1}{4}\begin{bmatrix} 5 & -3 \\ 3 & -5 \end{bmatrix}$$

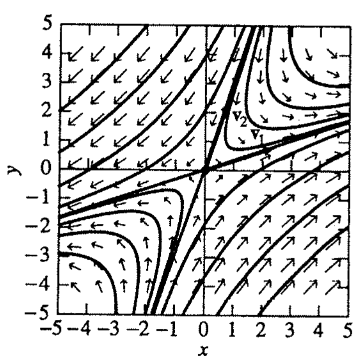


FIGURE 6.2.5. The saddle point of Example 3.

has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -1$ with associated eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 3 & 1 \end{bmatrix}^T$ and $\mathbf{v}_2 = \begin{bmatrix} 1 & 3 \end{bmatrix}^T$. Figure 6.2.5 shows a direction field and typical trajectories of the corresponding linear system $\mathbf{x}' = \mathbf{A}\mathbf{x}$. Note that the two eigenvectors again point in the directions of the linear trajectories. Here $k = -1$ and the nonlinear trajectories are hyperbolas in the oblique uv-coordinate system, so we have the saddle point indicated in the figure. Note that the two eigenvectors point in the directions of the asymptotes to these hyperbolas. ■

EQUAL REAL ROOTS: In this case, with $\lambda = \lambda_1 = \lambda_2 \neq 0$, the character of the critical point (0, 0) depends on whether or not the coefficient matrix **A** has two linearly independent eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$. If so, then we have oblique uv-coordinates as in Fig. 6.2.3, and the trajectories are described by

$$u(t) = u_0 e^{\lambda t}, \quad v(t) = v_0 e^{\lambda t} \tag{12}$$

as in (11). But now $k = \lambda_2/\lambda_1 = 1$, so the trajectories with $u_0 \neq 0$ are all of the form $v = Cu$ and hence lie on straight lines through the origin. Therefore, (0, 0) is a **proper node** (or **star**) as illustrated in Fig. 6.1.4, and is a source if $\lambda > 0$, a sink if $\lambda < 0$.

If the multiple eigenvalue $\lambda \neq 0$ has only a single associated eigenvector $\mathbf{v}_1$, then (as we saw in Section 5.4) there nevertheless exists a generalized eigenvector $\mathbf{v}_2$ such that $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_2 = \mathbf{v}_1$, and the linear system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ has the two linearly independent solutions

$$\mathbf{x}_1(t) = \mathbf{v}_1 e^{\lambda t} \quad \text{and} \quad \mathbf{x}_2(t) = (\mathbf{v}_1 t + \mathbf{v}_2)e^{\lambda t}. \tag{13}$$

We can still use the two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ to introduce oblique uv-coordinates as in Fig. 6.2.3. Then it follows from (13) that the coordinate functions $u(t)$ and $v(t)$ of the moving point $\mathbf{x}(t)$ on a trajectory are given by

$$u(t) = (u_0 + v_0 t)e^{\lambda t}, \quad v(t) = v_0 e^{\lambda t}, \tag{14}$$

where $u_0 = u(0)$ and $v_0 = v(0)$. If $v_0 = 0$ then this trajectory lies on the u-axis. Otherwise we have a nonlinear trajectory with

$$\frac{dv}{du} = \frac{dv/dt}{du/dt} = \frac{\lambda v_0 e^{\lambda t}}{v_0 e^{\lambda t} + \lambda(u_0 + v_0 t)e^{\lambda t}} = \frac{\lambda v_0}{v_0 + \lambda(u_0 + v_0 t)}.$$

We see that $dv/du \to 0$ as $t \to \pm\infty$, so it follows that each trajectory is tangent to the u-axis. Therefore, $(0, 0)$ is an **improper node.** If $\lambda < 0$, then we see from (14) that this node is a sink, but it is a source if $\lambda > 0$.

Example 4 The matrix

$$\mathbf{A} = \frac{1}{8}\begin{bmatrix} -11 & 9 \\ -1 & -5 \end{bmatrix}$$

has the multiple eigenvalue $\lambda = -1$ with the single associated eigenvector $\mathbf{v}_1 = \begin{bmatrix} 3 & 1 \end{bmatrix}^T$. It happens that $\mathbf{v}_2 = \begin{bmatrix} 1 & 3 \end{bmatrix}^T$ is a generalized eigenvector based on $\mathbf{v}_1$, but only the actual eigenvector shows up in a phase portrait for the linear system $\mathbf{x}' = \mathbf{A}\mathbf{x}$. As indicated in Fig. 6.2.6, the eigenvector $\mathbf{v}_1$ determines the u-axis through the improper nodal sink $(0, 0)$, this axis being tangent to each of the nonlinear trajectories. ■

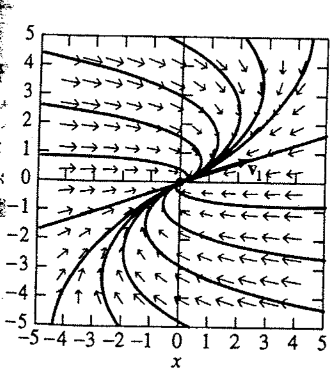


FIGURE 6.2.6. The improper nodal sink of Example 4.

Complex Conjugate Eigenvalues: Suppose that the matrix $\mathbf{A}$ has eigenvalues $\lambda = p + qi$ and $\bar{\lambda} = p - qi$ (with p and q both nonzero) having associated complex conjugate eigenvectors $\mathbf{v} = \mathbf{a} + \mathbf{b}i$ and $\bar{\mathbf{v}} = \mathbf{a} - \mathbf{b}i$. Then we saw in Section 5.2—see Eq. (22) there—that the linear system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ has the two independent real-valued solutions

$$\mathbf{x}_1(t) = e^{pt}(\mathbf{a}\cos qt - \mathbf{b}\sin qt) \quad \text{and} \quad \mathbf{x}_2(t) = e^{pt}(\mathbf{b}\cos qt + \mathbf{a}\sin qt). \tag{15}$$

Thus the components $x(t)$ and $y(t)$ of any solution $\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t)$ oscillate between positive and negative values as t increases, so the critical point $(0, 0)$ is a **spiral point** as in Example 5 of Section 6.1. If the real part p of the eigenvalues is negative, then it is clear from (15) that $\mathbf{x}(t) \to \mathbf{0}$ as $t \to +\infty$, so the origin is a spiral sink. But if p is positive, then the critical point is a spiral source.

Example 5 The matrix

$$\mathbf{A} = \frac{1}{4}\begin{bmatrix} -10 & 15 \\ -15 & 8 \end{bmatrix}$$

has the complex conjugate eigenvalues $\lambda = -\frac{1}{4} \pm 3i$ with negative real part, so $(0, 0)$ is a spiral sink. Figure 6.2.7 shows a direction field and a typical spiral trajectory approaching the origin as $t \to +\infty$. ■

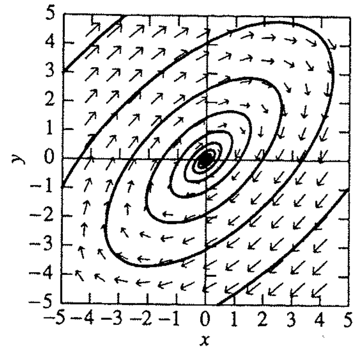


FIGURE 6.2.7. The spiral sink of Example 5.

PURE IMAGINARY EIGENVALUES: If the matrix **A** has conjugate imaginary eigenvalues $\lambda = qi$ and $\overline{\lambda} = -qi$ with associated complex conjugate eigenvectors $\mathbf{v} = \mathbf{a} + \mathbf{b}i$ and $\overline{\mathbf{v}} = \mathbf{a} - \mathbf{b}i$, then (15) with $p = 0$ gives the independent solutions

$$\mathbf{x}_1(t) = \mathbf{a}\cos qt - \mathbf{b}\sin qt \quad \text{and} \quad \mathbf{x}_2(t) = \mathbf{b}\cos qt + \mathbf{a}\sin qt \tag{16}$$

of the linear system $\mathbf{x}' = \mathbf{A}\mathbf{x}$. Just as in Example 4 of Section 6.1, it follows that any solution $\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t)$ describes an ellipse centered at the origin in the xy-plane. Hence (0, 0) is a **stable center** in this case.

Example 6 The matrix

$$\mathbf{A} = \frac{1}{4}\begin{bmatrix} -9 & 15 \\ -15 & 9 \end{bmatrix}$$

has the pure imaginary conjugate eigenvalues $\lambda = \pm 3i$, and therefore (0, 0) is a stable center. Figure 6.2.8 shows a direction field and typical elliptical trajectories enclosing the critical point. ■

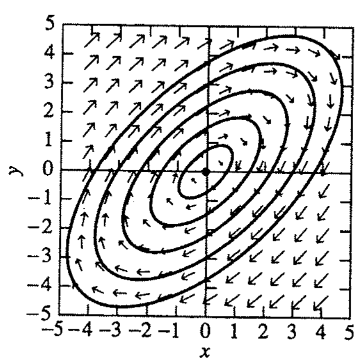


FIGURE 6.2.8. The stable center of Example 6.

Eigenvalues of A	Type of Critical Point
Real, unequal, same sign	Improper node
Real, unequal, opposite sign	Saddle point
Real and equal	Proper or improper node
Complex conjugate	Spiral point
Pure imaginary	Center

FIGURE 6.2.9. Classification of the critical point (0, 0) of the two-dimensional system $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

For the two-dimensional linear system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ with $\det \mathbf{A} \neq 0$, the table in Fig. 6.2.9 lists the type of critical point at (0, 0) found in the five cases discussed here, according to the nature of the eigenvalues λ_1 and λ_2 of the coefficient matrix **A**. Our discussion of the various cases shows that the stability of the critical point (0, 0) is determined by the *signs* of the real parts of these eigenvalues, as summarized in Theorem 1. Note that if λ_1 and λ_2 are real, then they are themselves their real parts.

THEOREM 1: Stability of Linear Systems

Let λ_1 and λ_2 be the eigenvalues of the coefficient matrix $\mathbf{A}$ of the two-dimensional linear system

$$\frac{dx}{dt} = ax + by,$$
$$\frac{dy}{dt} = cx + dy \tag{17}$$

with $ad - bc \neq 0$. Then the critical point $(0, 0)$ is

1. Asymptotically stable if the real parts of λ_1 and λ_2 are both negative;
2. Stable but not asymptotically stable if the real parts of λ_1 and λ_2 are both zero (so that $\lambda_1, \lambda_2 = \pm qi$);
3. Unstable if either λ_1 or λ_2 has a positive real part. ▲

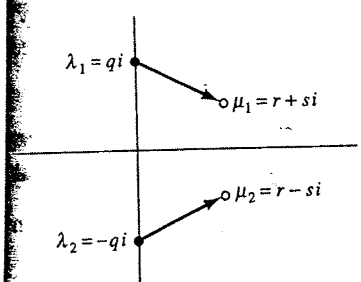


FIGURE 6.2.10. The effects of perturbation of pure imaginary roots.

It is worthwhile to consider the effect of small perturbations in the coefficients a, b, c, and d of the linear system in (17), which result in small perturbations of the eigenvalues λ_1 and λ_2. If these perturbations are sufficiently small, then positive real parts (of λ_1 and λ_2) remain positive and negative real parts remain negative. Hence an asymptotically stable critical point remains asymptotically stable and an unstable critical point remains unstable. Part 2 of Theorem 1 is therefore the only case in which arbitrarily small perturbations can affect the stability of the critical point $(0, 0)$. In this case pure imaginary roots $\lambda_1, \lambda_2 = \pm qi$ of the characteristic equation can be changed to nearby complex roots $\mu_1, \mu_2 = r \pm si$, with r either positive or negative (see Fig. 6.2.10). Consequently, a small perturbation of the coefficients of the linear system in (7) can change a stable center to a spiral point that is either unstable or asymptotically stable.

There is one other exceptional case in which the type, though not the stability, of the critical point $(0, 0)$ can be altered by a small perturbation of its coefficients. This is the case with $\lambda_1 = \lambda_2$, equal roots that (under a small perturbation of the coefficients) can split into two roots μ_1 and μ_2, which are either complex conjugates or unequal real roots (see Fig. 6.2.11). In either case, the sign of the real parts of the roots is preserved, so the stability of the critical point is unaltered. Its nature may change, however; the table in Fig. 6.2.9 shows that a node with $\lambda_1 = \lambda_2$ can either remain a node (if μ_1 and μ_2 are real) or change to a spiral point (if μ_1 and μ_2 are complex conjugates).

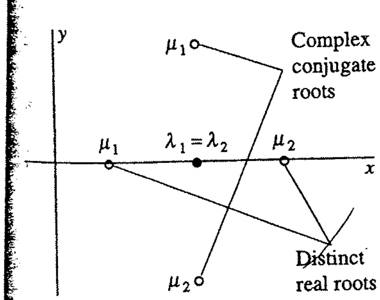


FIGURE 6.2.11. The effects of perturbation of real equal roots.

Suppose that the linear system in (17) is used to model a physical situation. It is unlikely that the coefficients in (17) can be measured with total accuracy, so let the unknown precise linear model be

$$\frac{dx}{dt} = a^\star x + b^\star y,$$
$$\frac{dy}{dt} = c^\star x + d^\star y. \tag{17*}$$

If the coefficients in (17) are sufficiently close to those in (17*), it then follows from the discussion in the preceding paragraph that the origin $(0, 0)$ is an asymptotically stable critical point for (17) if it is an asymptotically stable critical point for (17*), and is an unstable critical point for (17) if it is an unstable critical point for (17*).

Thus in this case the approximate model in (17) and the precise model in (17*) predict the same qualitative behavior (with respect to asymptotic stability versus instability).

Almost Linear Systems

We now return to the almost linear system

$$\begin{aligned}\frac{dx}{dt} &= ax + by + r(x, y),\\ \frac{dy}{dt} &= cx + dy + s(x, y)\end{aligned} \tag{18}$$

having (0, 0) as an isolated critical point with $ad - bc \neq 0$. Theorem 2, which we state without proof, essentially implies that—with regard to the type and stability of the critical point (0, 0)—the effect of the small nonlinear terms $r(x, y)$ and $s(x, y)$ is equivalent to the effect of a small perturbation in the coefficients of the associated *linear* system in (17).

THEOREM 2: Stability of Almost Linear Systems

Let λ_1 and λ_2 be the eigenvalues of the coefficient matrix of the linear system in (17) associated with the almost linear system in (18). Then

1. If $\lambda_1 = \lambda_2$ are equal real eigenvalues, then the critical point (0, 0) of (18) is either a node or a spiral point, and is asymptotically stable if $\lambda_1 = \lambda_2 < 0$, unstable if $\lambda_1 = \lambda_2 > 0$.
2. If λ_1 and λ_2 are pure imaginary, then (0, 0) is either a center or a spiral point, and may be either asymptotically stable, stable, or unstable.
3. Otherwise—that is, unless λ_1 and λ_2 are either real equal or pure imaginary—the critical point (0, 0) of the almost linear system in (18) is of the same type and stability as the critical point (0, 0) of the associated linear system in (17).

▲

Thus, if $\lambda_1 \neq \lambda_2$ and $\text{Re}(\lambda_1) \neq 0$, then the type and stability of the critical point of the almost linear system in (18) can be determined by analysis of its associated linear system in (17), and only in the case of pure imaginary eigenvalues is the stability of (0, 0) not determined by the linear system. Except in the sensitive cases $\lambda_1 = \lambda_2$ and $\text{Re}(\lambda_i) = 0$, the trajectories near (0, 0) will resemble qualitatively those of the associated linear system—they enter or leave the critical point in the same way, but may be "deformed" in a nonlinear manner. The table in Fig. 6.2.12 summarizes the situation.

An important consequence of the classification of cases in Theorem 2 is that *a critical point of an almost linear system is asymptotically stable if it is an asymptotically stable critical point of the linearization of the system.* Moreover, a critical point of the almost linear system is unstable if it is an unstable critical point of the linearized system. If an almost linear system is used to model a physical situation, then—apart from the sensitive cases mentioned earlier—it follows that the qualitative behavior of the system near a critical point can be determined by examining its linearization.

Eigenvalues λ_1, λ_2 for the Linearized System	Type of Critical Point of the Almost Linear System
$\lambda_1 < \lambda_2 < 0$	Stable improper node
$\lambda_1 = \lambda_2 < 0$	Stable node or spiral point
$\lambda_1 < 0 < \lambda_2$	Unstable saddle point
$\lambda_1 = \lambda_2 > 0$	Unstable node or spiral point
$\lambda_1 > \lambda_2 > 0$	Unstable improper node
$\lambda_1, \lambda_2 = a \pm bi \quad (a < 0)$	Stable spiral point
$\lambda_1, \lambda_2 = a \pm bi \quad (a > 0)$	Unstable spiral point
$\lambda_1, \lambda_2 = \pm bi$	Stable or unstable, center or spiral point

FIGURE 6.2.12. Classification of critical points of an almost linear system.

Example 7 Determine the type and stability of the critical point $(0, 0)$ of the almost linear system

$$\begin{aligned} \frac{dx}{dt} &= 4x + 2y + 2x^2 - 3y^2, \\ \frac{dy}{dt} &= 4x - 3y + 7xy. \end{aligned} \tag{19}$$

Solution The characteristic equation for the associated linear system (obtained simply by deleting the quadratic terms in (19)) is

$$(4 - \lambda)(-3 - \lambda) - 8 = (\lambda - 5)(\lambda + 4) = 0,$$

so the eigenvalues $\lambda_1 = 5$ and $\lambda_2 = -4$ are real, unequal, and have opposite signs. By our discussion of this case we know that $(0, 0)$ is an unstable saddle point of the linear system, and hence by Part 3 of Theorem 2, it is also an unstable saddle point of the almost linear system in (19). The trajectories of the linear system near $(0, 0)$ are shown in Fig. 6.2.13, and those of the nonlinear system in (19) are shown in Fig. 6.2.14. Figure 6.2.15 shows a phase portrait of the nonlinear system in (19) from a "wider view." In addition to the saddle point at $(0, 0)$, there are spiral points near the points $(0.279, 1.065)$ and $(0.933, -1.057)$, and a node near $(-2.354, -0.483)$. ■

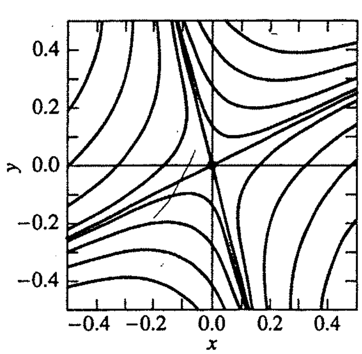


FIGURE 6.2.13. Trajectories of the linearized system of Example 7.

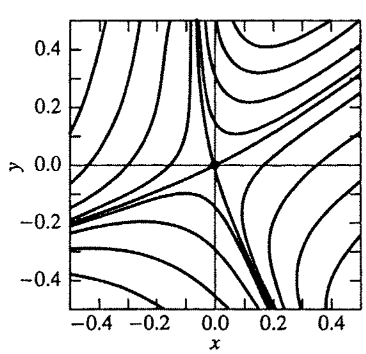


FIGURE 6.2.14. Trajectories of the original almost linear system of Example 7.

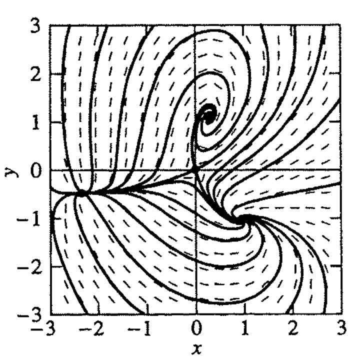


FIGURE 6.2.15. Phase portrait for the almost linear system in Eq. (19).

We have seen that the system $x' = f(x, y)$, $y' = g(x, y)$ with isolated critical point (x_0, y_0) transforms via the substitution $x = u + x_0$, $y = v + y_0$ to an equivalent uv-system with corresponding critical point $(0, 0)$ and linearization $\mathbf{u}' = \mathbf{J}\mathbf{u}$, whose coefficient matrix $\mathbf{J}$ is the Jacobian in (8) of the functions f and g at (x_0, y_0). Consequently we need not carry out the substitution explicitly; instead, we can proceed directly to calculate the eigenvalues of $\mathbf{J}$ preparatory to application of Theorem 2.

Example 8 Determine the type and stability of the critical point (4, 3) of the almost linear system

$$\frac{dx}{dt} = 33 - 10x - 3y + x^2,$$
$$\frac{dy}{dt} = -18 + 6x + 2y - xy. \tag{20}$$

Solution With $f(x, y) = 33 - 10x - 3y + x^2$, $g(x, y) = -18 + 6x + 2y - xy$ and $x_0 = 4$, $y_0 = 3$ we have

$$\mathbf{J}(x, y) = \begin{bmatrix} -10 + 2x & -3 \\ 6 - y & 2 - x \end{bmatrix}, \quad \text{so} \quad \mathbf{J}(4, 3) = \begin{bmatrix} -2 & -3 \\ 3 & -2 \end{bmatrix}.$$

The associated linear system

$$\frac{du}{dt} = -2u - 3v,$$
$$\frac{dv}{dt} = 3u - 2v \tag{21}$$

has characteristic equation $(\lambda + 2)^2 + 9 = 0$, with complex conjugate roots $\lambda = -2 \pm 3i$. Hence (0, 0) is an asymptotically stable spiral point of the linear system in (21), so Theorem 2 implies that (4, 3) is an asymptotically stable spiral point of the original almost linear system in (20). Figure 6.2.16 shows a typical trajectory of the linear system in (21), and Fig. 6.2.17 shows how this spiral point fits into the phase portrait for the original almost linear system in (20). ■

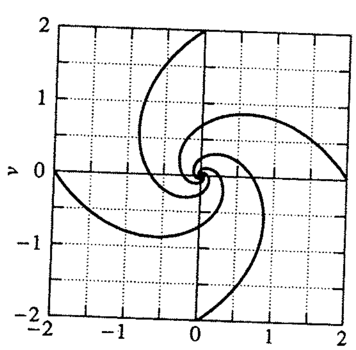


FIGURE 6.2.16. Spiral trajectories of the linear system in Eq. (21).

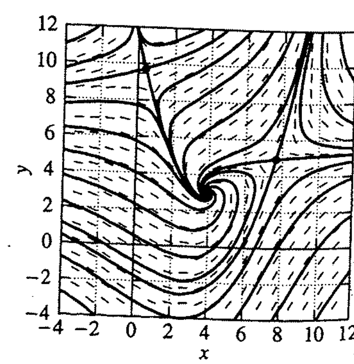


FIGURE 6.2.17. Phase portrait for the almost linear system in Eq. (20).

6.2 Problems

In Problems 1 through 10, apply Theorem 1 to determine the type of the critical point (0, 0) and whether it is asymtotically stable, stable, or unstable. Verify your conclusion by using a computer system or graphing calculator to construct a phase portrait for the given system.

1. $\frac{dx}{dt} = -2x + y, \quad \frac{dy}{dt} = x - 2y$

2. $\frac{dx}{dt} = 4x - y, \quad \frac{dy}{dt} = 2x + y$

3. $\frac{dx}{dt} = x + 2y, \quad \frac{dy}{dt} = 2x + y$

4. $\frac{dx}{dt} = 3x + y, \quad \frac{dy}{dt} = 5x - y$

5. $\frac{dx}{dt} = x - 2y, \quad \frac{dy}{dt} = 2x - 3y$

6. $\frac{dx}{dt} = 5x - 3y, \quad \frac{dy}{dt} = 3x - y$

7. $\frac{dx}{dt} = 3x - 2y, \quad \frac{dy}{dt} = 4x - y$

8. $\frac{dx}{dt} = x - 3y, \quad \frac{dy}{dt} = 6x - 5y$

9. $\frac{dx}{dt} = 2x - 2y, \quad \frac{dy}{dt} = 4x - 2y$

10. $\frac{dx}{dt} = x - 2y, \quad \frac{dy}{dt} = 5x - y$

Each of the systems in Problems 11 through 18 has a single critical point (x_0, y_0). Apply Theorem 2 to classify this critical point as to type and stability. Verify your conclusion by using a computer system or graphing calculator to construct a phase portrait for the given system.

11. $\frac{dx}{dt} = x - 2y, \quad \frac{dy}{dt} = 3x - 4y - 2$

12. $\frac{dx}{dt} = x - 2y - 8, \quad \frac{dy}{dt} = x + 4y + 10$

13. $\frac{dx}{dt} = 2x - y - 2, \quad \frac{dy}{dt} = 3x - 2y - 2$

14. $\frac{dx}{dt} = x + y - 7, \quad \frac{dy}{dt} = 3x - y - 5$

15. $\frac{dx}{dt} = x - y, \quad \frac{dy}{dt} = 5x - 3y - 2$

16. $\frac{dx}{dt} = x - 2y + 1, \quad \frac{dy}{dt} = x + 3y - 9$

17. $\frac{dx}{dt} = x - 5y - 5, \quad \frac{dy}{dt} = x - y - 3$

18. $\frac{dx}{dt} = 4x - 5y + 3, \quad \frac{dy}{dt} = 5x - 4y + 6$

In Problems 19 through 28, investigate the type of the critical point (0, 0) of the given almost linear system. Verify your conclusion by using a computer system or graphing calculator to construct a phase portrait. Also, describe the approximate locations and apparent types of any other critical points that are visible in your figure. Feel free to investigate these additional critical points; you can use the computational methods discussed in the application material for this section.

19. $\frac{dx}{dt} = x - 3y + 2xy, \quad \frac{dy}{dt} = 4x - 6y - xy$

20. $\frac{dx}{dt} = 6x - 5y + x^2, \quad \frac{dy}{dt} = 2x - y + y^2$

21. $\frac{dx}{dt} = x + 2y + x^2 + y^2, \quad \frac{dy}{dt} = 2x - 2y - 3xy$

22. $\frac{dx}{dt} = x + 4y - xy^2, \quad \frac{dy}{dt} = 2x - y + x^2y$

23. $\frac{dx}{dt} = 2x - 5y + x^3, \quad \frac{dy}{dt} = 4x - 6y + y^4$

24. $\frac{dx}{dt} = 5x - 3y + y(x^2 + y^2), \quad \frac{dy}{dt} = 5x + y(x^2 + y^2)$

25. $\frac{dx}{dt} = x - 2y + 3xy, \quad \frac{dy}{dt} = 2x - 3y - x^2 - y^2$

26. $\frac{dx}{dt} = 3x - 2y - x^2 - y^2, \quad \frac{dy}{dt} = 2x - y - 3xy$

27. $\frac{dx}{dt} = x - y + x^4 - y^2, \quad \frac{dy}{dt} = 2x - y + y^4 - x^2$

28. $\frac{dx}{dt} = 3x - y + x^3 + y^3, \quad \frac{dy}{dt} = 13x - 3y + 3xy$

In Problems 29 through 32, find all critical points of the given system, and investigate the type and stability of each. Verify your conclusions by means of a phase portrait constructed using a computer system or graphing calculator.

29. $\frac{dx}{dt} = x - y, \quad \frac{dy}{dt} = x^2 - y$

30. $\frac{dx}{dt} = y - 1, \quad \frac{dy}{dt} = x^2 - y$

31. $\frac{dx}{dt} = y^2 - 1, \quad \frac{dy}{dt} = x^3 - y$

32. $\frac{dx}{dt} = xy - 2, \quad \frac{dy}{dt} = x - 2y$

Bifurcations

The term bifurcation *generally refers to something "splitting apart." With regard to differential equations or systems involving a parameter, it refers to abrupt changes in the character of the solutions as the parameter is changed continuously. Problems 33 through 36 illustrate sensitive cases in which small perturbations in the coefficients of a linear or almost linear system can change the type or stability (or both) of a critical point.*