
Differential Equations

Math 341 Fall 2008
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MWF 2:30-3:25pm Fowler 307
<http://faculty.oxy.edu/ron/math/341/08/>

Worksheet 29: Monday November 17

TITLE The Laplace Transform and Second Order ODEs

CURRENT READING Blanchard, 6.3

Homework Assignments due Monday November 24

Section 6.3: 5, 6, 8, 9, 10, 15, 18, 27, 28.

Section 6.4: 1, 2, 5, 7, 8.

Section 6.5: 2, 5, 6, 9

SUMMARY

We shall learn how to apply Laplace Transforms to solve second-order ordinary differential equations of the form $y'' + py' + qy = f(t)$ and especially ones that have a brand-new wild and wacky mathematical object called the Dirac Delta Function. We shall also become more familiar with Mathematica.

RECALL Zill, Example 3, page 295.

Let's use Mathematica to show that the solution of $y'' - 6y' + 9y = t^2 e^{3t}$, $y(0) = 2$, $y'(0) = 17$ is $y(t) = 2e^{3t} + 11te^{3t} + \frac{1}{12}t^4 e^{3t}$.

1. Using Mathematica to Solve ODEs

First of all, Mathematica can be used to solve this ODE directly.

The command to use is DSolve. Specifically

```
DSolve[ {y''[t]-6y'[t]+9y[t]==t^2 Exp[ 3 t ],y[0]==2, y'[0]==17 },y[t],t]
```

Type the above command verbatim and press SHIFT-ENTER to look at the results!

We can also use Mathematica to find Laplace Transforms that we need to solve the problem.

```
LaplaceTransform[{y''[t]-6y'[t]+9y[t]==t^2 Exp[ 3 t ]},t,s]
```

```
Solve[%, LaplaceTransform[y[t], t, s]]
```

```
InverseLaplaceTransform[%, s, t]
```

So, the commands to remember are DSolve, InverseLaplaceTransform[F[s], s, t] and LaplaceTransform[y[t], t, s].

NOTE the different order of s and t in the two commands!

Also, the Heaviside Function in Mathematica is called HeavisideTheta[x].

2. Applications of Laplace Transforms to Linear Second-Order ODEs

GROUPWORK

Solve the following initial value problems using Laplace Transforms (and Mathematica)! **Zill,**

page 303, #31. $y'' + y = f(t)$, $y(0) = 0$, $y'(0) = 1$ where $f(t) = \begin{cases} 0, & 0 \leq t < \pi \\ 1, & \pi \leq t < 2\pi \\ 0, & 2\pi \leq t \end{cases}$

Blanchard, page 594, #29. $y'' - 4y' + 5y = 2e^t$, $y'(0) = 1$, $y(0) = 1$.

3. The Unit Impulse Function

Consider the unit impulse function $\delta_a(t) = \begin{cases} 0, & 0 \leq t < t_0 - a \\ \frac{1}{2a}, & t_0 - a < t < t_0 + a \\ 0, & t_0 + a < t \end{cases}$

DEFINITION: Dirac Delta Function The **Dirac Delta Function** is denoted by $\delta(t-t_0)$ and is the object (it's not really a function) which results when one takes the limit as $a \rightarrow 0$ of the unit impulse function $\delta_a(t-t_0)$. In other words, $\delta(t-t_0) = \begin{cases} 0, & t \neq t_0 \\ \infty, & t = t_0 \end{cases}$.

The Dirac Delta Function also has the property that $\int_{-\infty}^{\infty} \delta(t-t_0) dt = 1$

THEOREM: The Laplace Transform of the Dirac Delta Function

For $t_0 > 0$, $\mathcal{L}[\delta(t-t_0)] = e^{-st_0}$ and $\mathcal{L}^{-1}[e^{-st_0}] = \delta(t-t_0)$. (For more details, see Blanchard, p. 597).

Interestingly, we can relate the Heaviside function $\mathcal{H}(t)$ and Dirac Delta Function $\delta(t)$. Consider the following integrally defined function $f(x) = \int_{-\infty}^x \delta(t-t_0) dt$.

Q: What does $f(x)$ look like?

A: Depends on the relationship between x and t_0 . How? Can you draw a picture of it?

The integral of the _____ is the _____, and the _____ of Heaviside Function is equal to the Dirac Delta Function. (Pretty cool, eh?) Sketch the Heaviside Function and Dirac Delta Function for all t values.

Also, the Delta Function in Mathematica is called `DiracDelta[x]`.

4. Delta Function as Source Term

What's interesting about the Dirac Delta Function is that it allows us to model situations where an instantaneous impulse is applied to a system at a certain time. Laplace Transforms are really the only technique which allow solution of such initial value problems.

EXAMPLE

Zill, page 316, Example 1. Solve $y'' + y = 4\delta(t - 2\pi)$ where

(a) $y(0) = 1, \quad y'(0) = 0$ and (b) $y(0) = 0, \quad y'(0) = 0$

Exercise

In the space below, sketch the solutions to the initial value problems from the previous example, i.e. $y'' + y = 4\delta(t - 2\pi), \quad y(0) = 1, y'(0) = 0$ and

$y'' + y = 4\delta(t - 2\pi), \quad y(0) = 0, y'(0) = 0$