Differential Equations

Math 341 Fall 2008 MWF 2:30-3:25pm Fowler 307

 `http://faculty.oxy.edu/ron/math/341/08/`

Worksheet 23: **Wednesday October 29**

TITLE Qualitative Analysis of Nonlinear Systems
CURRENT READING Blanchard, 5.2

Homework Assignments due Friday October 31
Section 3.7: 1, 6.
Section 5.1: 3, 4, 5, 18, 21.
Section 5.2: 3, 4, 16.

SUMMARY
We shall explore how to analyze phase portraits of systems without using numerics or linearization. We'll also explore bifurcations of quasi-linear systems.

1. Two-Species Competition Model

Consider the following non-linear systes of ODEs

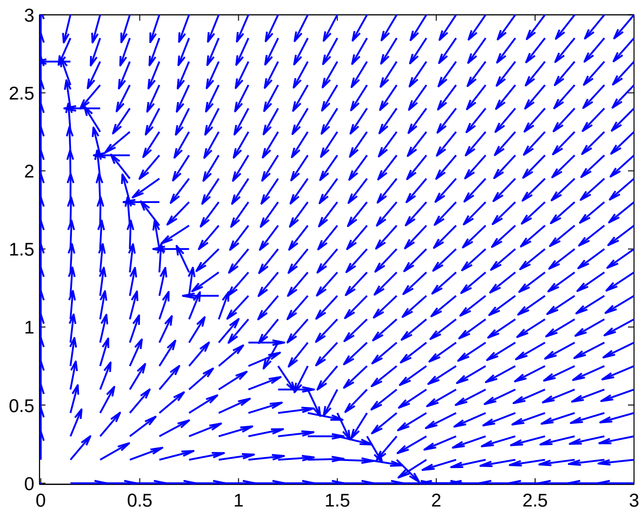
$$\frac{dx}{dt} = 2x\left(1-\frac{x}{2}\right) - xy$$
$$\frac{dy}{dt} = 3y\left(1-\frac{y}{3}\right) - 2xy$$

Exercise

Show that the equilibrium points are $(0,0)$, $(0,3)$, $(2,0)$ and $(1,1)$ and use the linearization technique to classify them and describe their stabiliy.

Here is the direction field for

$$\begin{aligned}\frac{dx}{dt} &= 2x\left(1-\frac{x}{2}\right) - xy\\ \frac{dy}{dt} &= 3y\left(1-\frac{y}{3}\right) - 2xy\end{aligned}$$



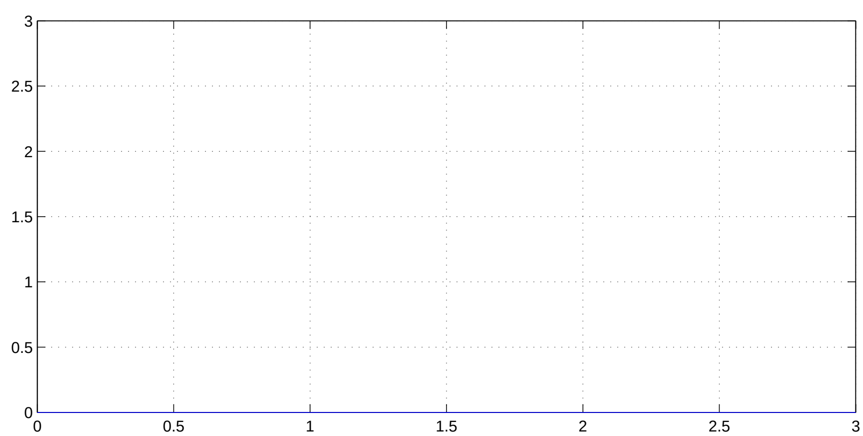
2. Qualitative Analysis

What we'd like to do is understand the long-term behavior of solutions to the above system. Where do solutions end up for the vast majority of possible initial conditions in the plane? NOTE: this corresponds to trying to determine which of the two species in competition, species A (corresponding to $x(t)$) or species B (corresponding to $y(t)$) "wins" or whether the two species can co-exist.

RECALL

On *Worksheet 11* we defined the term **nullcline**. An x-nullcline is a set of points for which x is constant (i.e. $\frac{dx}{dt} = 0$). Algebraically, $\{(x, y) \mid f(x, y) = 0\}$. A y-nullcline is a set of points for which y is constant (i.e. $\frac{dy}{dt} = 0$). In this case, $\{(x, y) \mid g(x, y) = 0\}$.

On the set of axes below draw in the x-nullclines and y-nullclines of the given linear system of ODEs

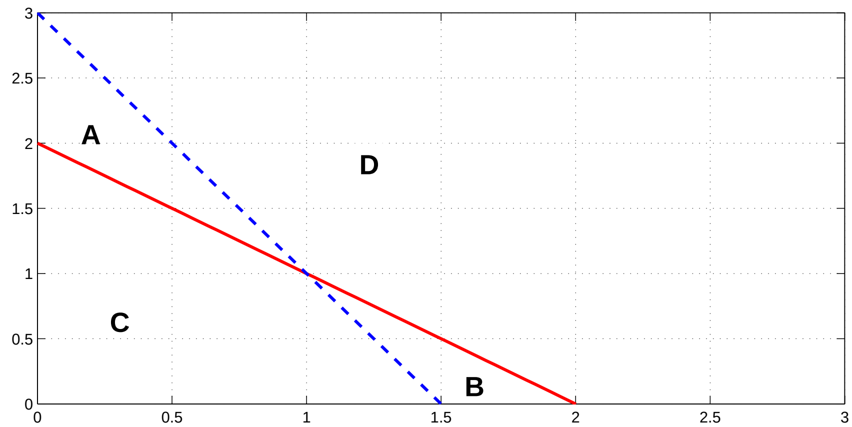


3. Nullcline Analysis

Clearly indicate the regions in which $\dfrac{dx}{dt}$ and $\dfrac{dy}{dt}$ are either positive or negative. There are four possibilities

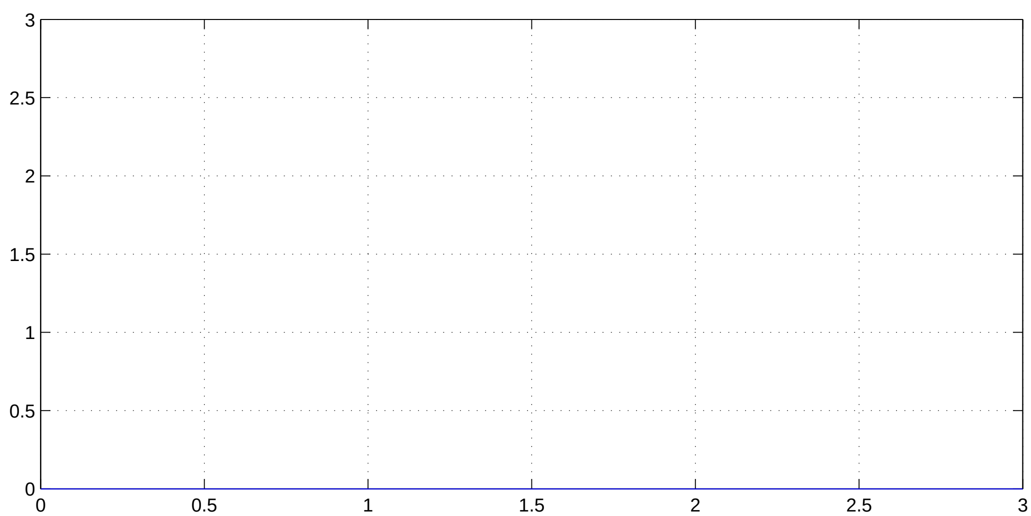
- $\frac{dy}{dt} > 0$ and $\frac{dx}{dt} > 0$ which means solutions are moving **UP** and **RIGHT**
- $\frac{dy}{dt} < 0$ and $\frac{dx}{dt} > 0$ which means solutions are moving **DOWN** and **RIGHT**
- $\frac{dy}{dt} > 0$ and $\frac{dx}{dt} < 0$ which means solutions are moving **UP** and **LEFT**
- $\frac{dy}{dt} < 0$ and $\frac{dx}{dt} < 0$ which means solutions are moving **DOWN** and **LEFT**

We notice the nullclines split the plane into four regions labelled A, B, C and D.



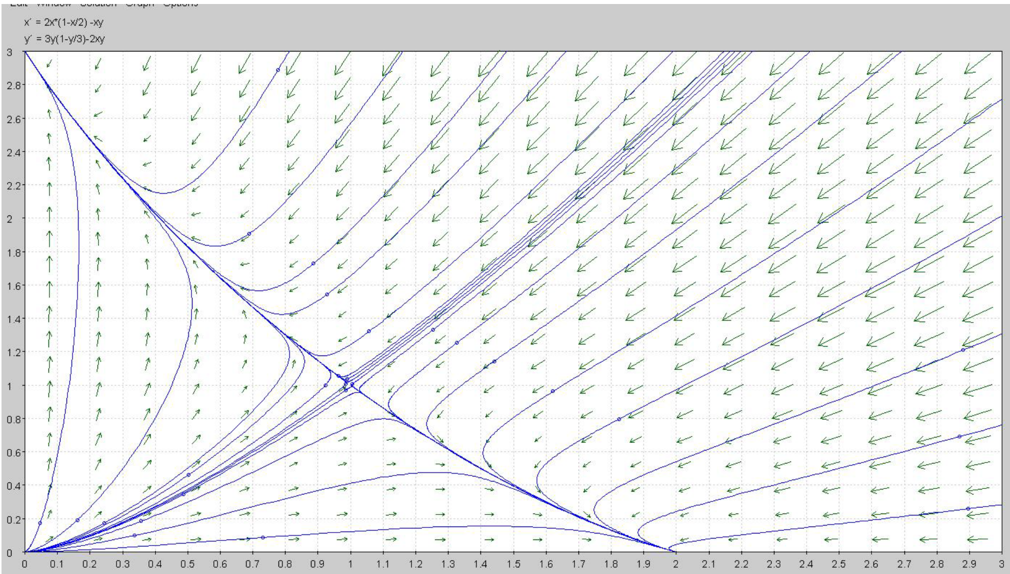
$\boxed{\text{EXAMPLE}}$
On the figure, label the regions accordingly, using one of the four possibilities listed above. That allows us to be able to tell which regions of the phase portrait tend towards which equilibrium points in the long term.

We can use all this information (including what we know about the equilibrium points from our linearization analysis) to sketch the phase portrait on the axes below.



Here is the actual phase portrait for the system

$$\begin{aligned}\frac{dx}{dt} &= 2x\left(1-\frac{x}{2}\right) - xy\\ \frac{dy}{dt} &= 3y\left(1-\frac{y}{3}\right) - 2xy\end{aligned}$$



4. Linearization Doesn't Always Work!

There are at least two cases where linearization can lead one astray. If the linearized system has a **center** as a stationary point it is possible for the non-linear system to be slightly different and that equilibrium to actually be a spiral source or sink instead.

Also, if the linearized system has zero as an eigenvalue it is possible that the nonlinear system may exhibit different behavior.

For example, consider

$$\begin{aligned}\frac{dx}{dt} &= y + ax(x^2+y^2)\\ \frac{dy}{dt} &= -x + ay(x^2+y^2)\end{aligned}$$

where a is some parameter.

GroupWork

Show that the linearized system has a center but for values of $a > 0$ the nonlinear system has a spiral source and for $a < 0$ it has a spiral sink at the origin.