# Differential Equations

Math 341 Fall 2008 MWF 2:30-3:25pm Fowler 307

http://faculty.oxy.edu/ron/math/341/08/

## *Worksheet 22*: **Monday October 27**

**TITLE** Linearization
**CURRENT READING** Blanchard, 5.1

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**Homework Assignments due Friday October 31**
Section 3.7: 1, 6.
Section 5.1: 3, 4, 5, 18, 21.
Section 5.2: 3, 4, 16.

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**SUMMARY**
We shall begin our analysis of non-linear systems using a technique called linearization which transforms the behavior of nonlinear systems of ODEs back into our now familiar analysis of linear systems of ODEs. Remember your Taylor Aproximations!

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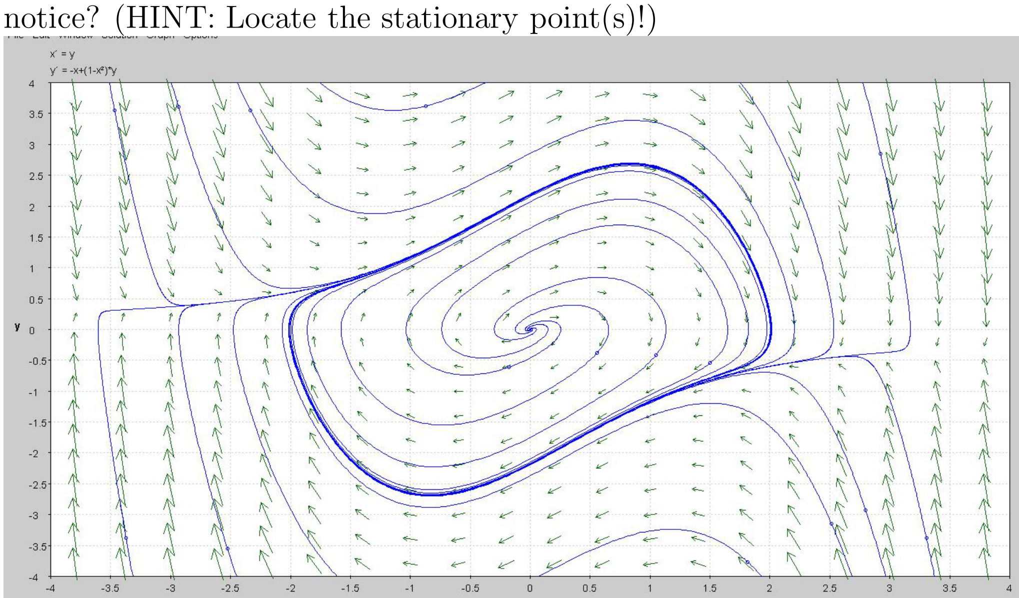
## 1. The Van der Pol Equation

An important nonlinear system of ODEs which occurs in Physics is the Van der Pol Equation for $x(t)$ $x'' + x - (1-x^2)x' = 0$ which can be written as a non-linear system as

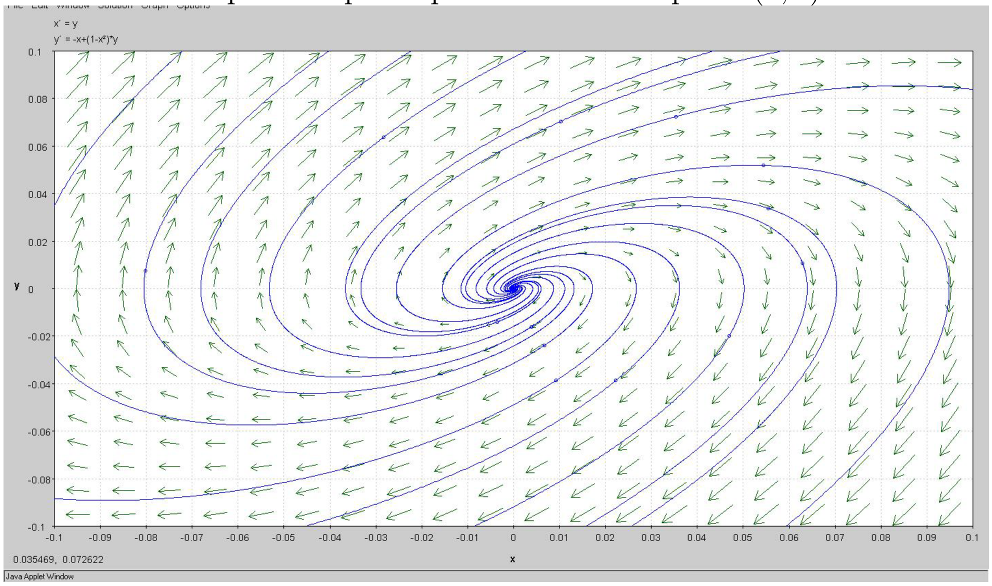
$$\begin{aligned} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= -x + (1-x^2)y \end{aligned}$$

Below is the direction field and phase portrait for the Van der Pol system. What do you notice? (HINT: Locate the stationary point(s)!)




**Q:** What happens to solutions that start near the origin at (0,0)? What about solutions that start (relatively) far away at (3,3)?
**A**:

Here is a close up of the phase portrait near the point $(0, 0)$




**Q:** What can we say about the stationary point at (0,0) of the Van der Pol system?
**A:** ________________________________________

**EXAMPLE**

Let's use the technique of linearization to explain the behavior near the origin of the Van der Pol system. Suppose $x$ and $y$ are close to 0.1 in size, then the nonlinear term $x^2y$ will be close to $(0.1^3)$ in magnitude, much ________________________ then either $x$ or $y$.

We can therefor write a linearized version of the Van der Pol system which looks like

$$\begin{aligned} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= -x + y \end{aligned}$$

which when written as a matrix looks like $\frac{d\vec{x}}{dt} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \vec{x}$ where $\vec{x} = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$

**Exercise**

Find the eigenvalues of the linearized Van der Pol system and use this information to classify the stationary point at (0,0).

## 2. The Linearization Process

**RECALL**

**Definition: Jacobian matrix**

The **derivative matrix** (usually called the **Jacobian**) of a vector function $\vec{f} : \mathbb{R}^n \to \mathbb{R}^m$ is the matrix consisting of the $n$ partial derivatives of each of the $m$ co-ordinate functions arranged so that the rows of the matrix are exactly gradient vectors of each coordinate function. The Jacobian has $mn$ entries where $J_{i,j} = \dfrac{\partial f_i}{\partial x_j}$. In other words,

$$J(\vec{x}) = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} & \cdots & \dfrac{\partial f_1}{\partial x_n} \\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} & \cdots & \dfrac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial f_m}{\partial x_1} & \dfrac{\partial f_m}{\partial x_2} & \cdots & \dfrac{\partial f_m}{\partial x_n} \end{bmatrix}$$

Consider the general form of a 2-dimensional nonlinear system of 1st order ODEs

$$\begin{aligned} \frac{dx}{dt} &= f(x, y) \\ \frac{dy}{dt} &= g(x, y) \end{aligned}$$

This can also be thought of as $\dfrac{d\vec{x}}{dt} = \vec{f}(\vec{x})$. Clearly in this case $f : \mathbb{R}^2 \to \mathbb{R}^2$ so the Jacobian matrix $J$ for $\vec{f}(\vec{x})$ would be a ________________________________________.

We can always use Taylor's Theorem for Vector-Valued Functions to approximate the function $\vec{f}(\vec{x})$ near a point $\vec{x}_0$ by saying

$$\vec{f}(\vec{x}) \approx \vec{f}(\vec{x}_0) + J(\vec{x}_0)(\vec{x} - \vec{x}_0) + \ldots$$

This will be extrememly useful if the nonlinear system has a fixed point at the point $(x_0, y_0)$ (also known as $\vec{x}_0$) because then we will be able to analyze a linear system of the form

$$\frac{d\vec{x}}{dt} = J(\vec{x}_0)(\vec{x} - \vec{x}_0)$$

instead of the original nonlinear system

In fact, usually the change of variables $\vec{u} = \vec{x} - \vec{x}_0$ will be made and we will be analyzing the system

$$\frac{d\vec{u}}{dt} = J(x_0, y_0)\vec{u}, \qquad \text{where } \vec{u} = \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}$$

where the fixed point will now be at the origin of the $(u, v)$-system instead of at $(x_0, y_0)$ in the $xy$-plane.

GROUPWORK

Consider

$$\begin{aligned}\frac{dx}{dt} &= -x + x^3\\ \frac{dy}{dt} &= -2y\end{aligned}$$

Identify and then classify all the equilibria of the non-linear system of ODEs, using the Linearization Process. (HINT: calculate the Jacobian, evaluate at each equilibria, compute the eigenvalues and classify the equilibria)