Differential Equations

Math 341 Fall 2008 — MWF 2:30-3:25pm Fowler 307

 — `http://faculty.oxy.edu/ron/math/341/08/`

Worksheet 21: **Friday October 24**

TITLE The Trace-Determinant Plane
CURRENT READING Blanchard, 3.7

Homework Assignments due Friday October 31
Section 3.7: 1, 6.
Section 5.1: 3, 4, 5, 18, 21.
Section 5.2: 3, 4, 16.

SUMMARY
We shall summarize all the possible equilibria one can get with a 2x2 linear system of ODEs into one big picture!

1. Summarizing The Possibilities

Given a system of linear ODEs with associated matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ the characteristic polynomial is $(a-\lambda)(d-\lambda) - bc = \lambda^2 - (a+d)\lambda + ad - bc = \lambda^2 - \text{tr(A)}\lambda + \det(\text{A}) = 0$.

GroupWork

Your goal is to match the case # in the left column with the description of its critical point on the right (the list now is jumbled).

CASE 1: Real λ, $\lambda_1\lambda_2 < 0$	**A** Center
CASE 2: Real λ, $\lambda_1 \& \lambda_2 < 0$	**B** Spiral Source
CASE 3: Real λ, $\lambda_1 \& \lambda_2 > 0$	**C** (Stable) Node
CASE 4: Real λ, $\lambda_1 = \lambda_2 > 0$	**D**(Unstable) Node
CASE 5: Real λ, $\lambda_1 = \lambda_2 < 0$	**E** Saddle
CASE 6: Complex λ, $\text{Re}(\lambda) > 0$	**F** Spiral Sink
CASE 7: Complex λ, $\text{Re}(\lambda) < 0$	**G** Sink
CASE 8: Complex λ, $\text{Re}(\lambda) = 0$	**H** Source

Run the CD-Rom from our textbook and select `LinearPhasePortaits`. Use the slide bars to obtain different values of a, b, c and d and the different kinds of eigenvalues recorded above in the Cases. Record your results in the table below.

CASE #	a	b	c	d	λ_1	λ_2	Description
1							
2							
3							
4							
5							
6							
7							
8							

For more details, see the handout from **Edwards and Penney,** *Differential Equations*, **3rd Edition, Prentice Hall: 2004.**

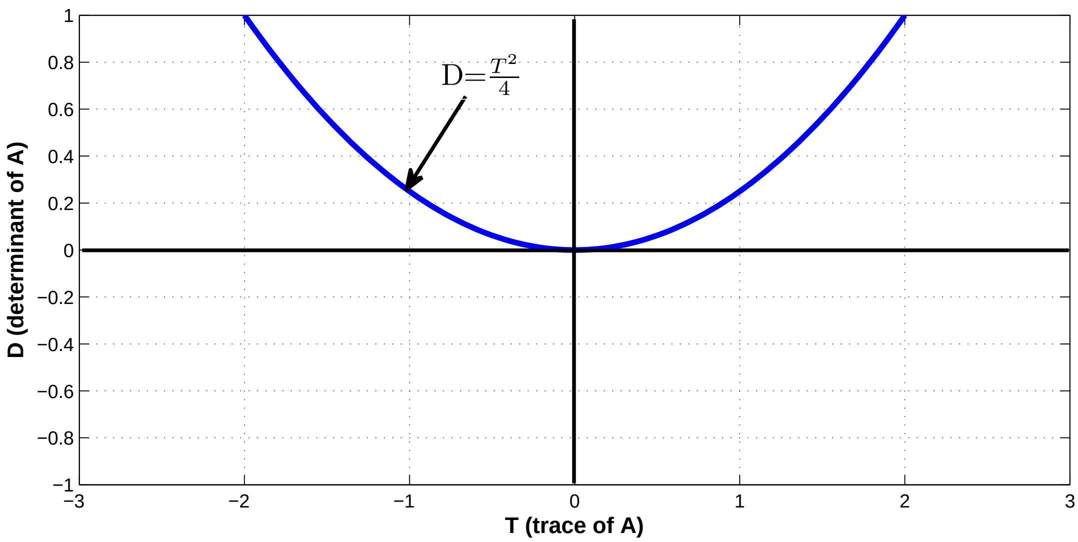
2. The Trace-Determinant Plane

Recall that the eigenvalues of a 2x2 matrix are given by the rootsof the polynomial $p(\lambda) = \lambda^2 - \text{tr}(A) + \det(A) = 0$.

It's also true that the trace of A, denoted $\text{tr}(A)$ is equal to the **sum** of the eigenvalues $\lambda_1 + \lambda_2$. Let's use the symbol T for $\text{tr}(A)$. The determinant of A, denoted $\det(A)$ is equal to the **product** of the eigenvalues $\lambda_1\lambda_2$. Let's use the symbol D for $\det(A)$.

Then we know that the eigenvalues are given by the solutions to $\lambda^2 - T\lambda + D = 0$, or $\lambda = \dfrac{T \pm \sqrt{T^2 - 4D}}{2}$.

In other words, the condition on whether we will have real, complex or repeated eigenvalues depends on the behavior of the discriminant $\Gamma = T^2 - 4D$. See the figure drawn below. This is known as the **Trace-Determinant Plane**



This graph is an example of a parameter plane. As the matrix A changes it has different values of T and D and the linear system $\dfrac{d\vec{x}}{dt} = A\vec{x}$ corresponding to that matrix will be located at a diferent location in (T, D)-space.

Exercise

(1) What kind of phase portraits will exist in (T, D)-space along the D axis?

(2) What about the T-axis?

(3) What kind of phase portraits occur along the curve $D = \frac{T^2}{4}$?

(4) What happens as one moves from the region just above the T-axis $(D > 0)$ to just below the T-axis $(D > 0)$? Does it matter if $T >)$ or $T < 0$?

(5) What kinds of solutions exist in the region above the parabola $D = \frac{T^2}{4}$?