Differential Equations

Math 341 Fall 2008 MWF 2:30-3:25pm Fowler 307

 http://faculty.oxy.edu/ron/math/341/08/

Worksheet 11: Wednesday September 24

TITLE Geometry of First Order Systems of ODEs
CURRENT READING Blanchard, 2.2

Homework Assignments due Friday September 26
Section 1.9: 4, 5, 9, 12, 19.
Chapter 1 Review: 3, 4, 6, 10, 11, 12, 29, 47.
Section 2.1: 1, 2, 3, 5, 7, 10.

SUMMARY
We will learn how to create the beautiful pictures which can result when one does quantitative analysis on systems of ODEs (phase portraits).

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1. Vector Notation and Vector Fields

Let $\vec{x} = \begin{bmatrix} R(t) \\ F(t) \end{bmatrix}$ and $\dfrac{d\vec{x}}{dt} = \begin{bmatrix} R'(t) \\ F'(t) \end{bmatrix}$, the Lotka-Volterra equations can be re-written as:

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} aR - bRF \\ cF + dRF \end{bmatrix} = \vec{P}(\vec{x}, t)$$

Note that the above $\vec{P}$ is a vector function of a vector input, in the case of Lotka-Volterra $P : \mathbb{R}^3 \to \mathbb{R}^2$

DEFINITION: **fixed point of linear system of ODEs**

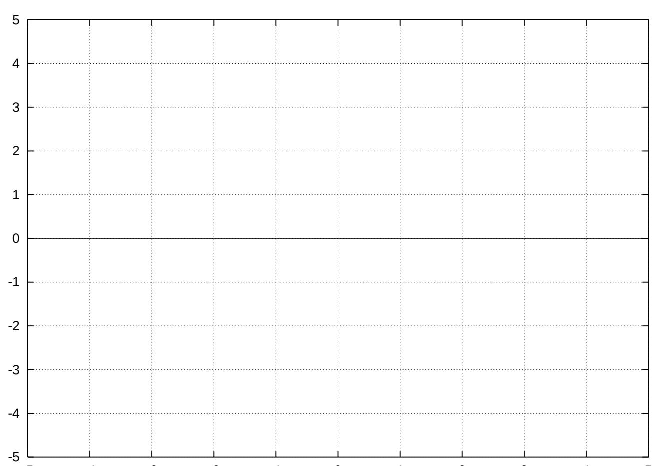
A fixed point or equilibrium point or stationary point $\vec{x_0}$ of the system $\dfrac{d\vec{x}}{dt} = \vec{F}(\vec{x}, t)$ is a point at which $\vec{F}(\vec{x_0}) = \vec{0}$.

RECALL

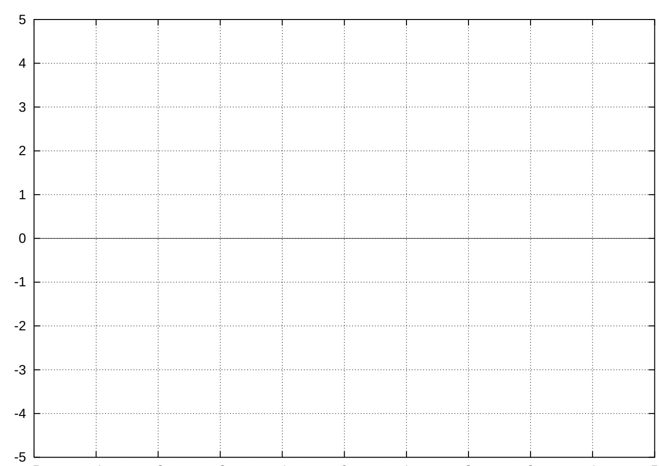
We can visualize vector functions using **vector fields**. Consider the function $\vec{F}(x, y) = \begin{bmatrix} y \\ -x \end{bmatrix}$. Sketch the vector field in the axes to the left, below. Generally, we normalize the vectors to all have the same magnitude and produce something that is called a **direction field**. It looks exactly like a **slope field**, except the "lineal elements" have little arrows on them.

EXAMPLE Let's draw the vector field and direction field for $\vec{F}(\vec{x}) = (y, -x)$.

Vector Field for $\vec{F} = (y, -x)$



Direction Field for $\vec{F} = (y, -x)$



2. Direction Fields and 1st Order Linear Systems of ODEs

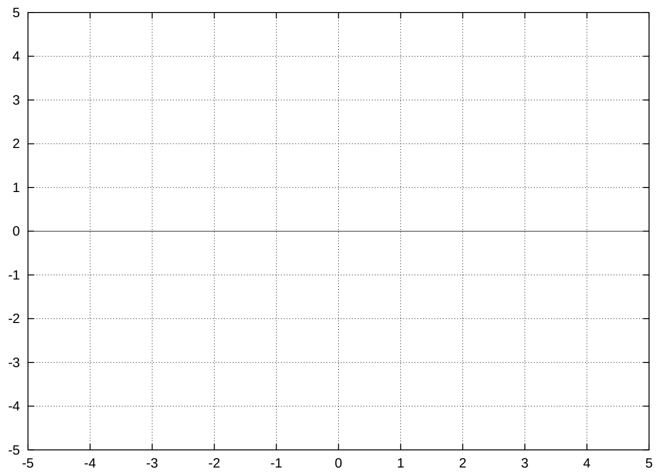
Consider the system of ODES

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = -x$$

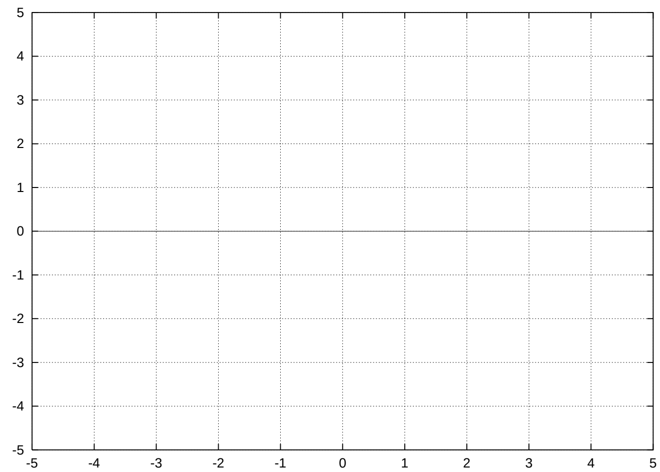
GROUPWORK

Draw a solution curve that starts at $x(0) = 0, \quad y(0) = 1$ on the axes to the left, and on the right draw solution curves of $x(t)$ and $y(t)$ on the same axes.

Solution Curve for $\vec{x}(0) = (0, 1)$



$x(t)$ and $y(t)$ versus t



Q: Does the system have any equilibria?
A: ____

3. Nullclines

Consider the general 2-D system of ODEs

SYSTEM A

$$\frac{dx}{dt} = f(x, y)$$
$$\frac{dy}{dt} = g(x, y)$$

DEFINITION: **nullcline**

A curve along which a derivative (with respect to the independent variable) is zero is said to be a nullcline. In other words, one of the variables will be constant, while the other variable varies with respect to t. An x-nullcline is a set of points for which x is constant (i.e. $\frac{dx}{dt} = 0$). Algebraically, $\{(x, y) \mid f(x, y) = 0\}$. A y-nullcline is a set of points for which y is constant (i.e. $\frac{dy}{dt} = 0$). In this case, $\{(x, y) \mid g(x, y) = 0\}$.

Q: What are the nullclines of System A?
A: ____

4. Phase Portrait

The phase portrait of a system is a diagram showing the set of solution curves in the phase plane of a system of ODEs.

SYSTEM B

$$\frac{dx}{dt} = 5x$$
$$\frac{dy}{dt} = -y$$

Exercise

Sketch the phase portrait of the system. This means sketching the nullclines, clearly indicating any fixed points and including several solution curves in the phase plane.

SYSTEM C

$$\frac{dx}{dt} = x + y$$
$$\frac{dy}{dt} = 4x - 2y$$

Exercise

Sketch the phase portrait of the system. This means sketching the nullclines, clearly indicating any fixed points and including several solution curves in the phase plane.

SYSTEM D

$$\begin{aligned} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= -\frac{k}{m}x \end{aligned}$$

This system should look familar, or if it doesn't perhaps this differential equation is

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0 \tag{1}$$

The above equation is the equation of motion for **harmonic motion** and has the known solutions $x(t) = A\cos(\omega t) + B\sin(\omega t)$ where $\omega^2 = \dfrac{k}{m}$ and ω is the frequency of the motion and $\dfrac{2\pi}{\omega}$ is the period of the oscillation of a mass on a string (with no damping).

What's the relationship between Equation 1 and System D? Is there a way to convert one into the other?

GROUPWORK

Use technology to sketch phase portraits of System D for various values of the ratio k/m.