Differential Equations

 http://faculty.oxy.edu/ron/math/341/08/

Worksheet 7: **Monday September 15**

TITLE Introduction to Bifurcations
CURRENT READING Blanchard, 1.7

Homework Assignments due Friday September 19
Section 1.7: 3, 6, 8, 12, 15.
Section 1.8: 4, 5, 8, 9, 17, 18, 20.

SUMMARY
We will learn about a modern analytical technique which allows one to analyze differential equations which contain parameters.

1. Parameter Sensitivity

Consider a model for logistic growth of a fish population with constant harvesting given by the IVP $P' = P(5 - P) - h, \quad P(0) = P_0$ where $h \geq 0$. Let's investigate how or if the solution changes as the values of the parameter h changes.

GROUPWORK

In the space below, draw phase lines for the critical points of the above IVP when the value of h equals 0, 2, 4, 6 and 8. Identify and classify any and all critical points for each value of h. What do you notice?

Is there a particular value of h for which the nature of the solution changes? If so, find it.

DEFINITION: **bifurcation**

A change in the qualitative nature of the phase portrait or long-term behavior of the solution of a differential equation when the value of a parameter changes is called a **bifurcation** of the DE. The value at which such changes occur is known as a **bifurcation point** or **bifurcation value** of the DE.

DEFINITION: **hyperbolic and nonhyperbolic critical points**

A critical point of an autonomous DE $y' = f(y)$ is said to be **nonhyperbolic** if arbitrarily small changes (known as **perturbations**) in $f(y)$ cause a bifurcation in the DE, i.e. critical points appear or disappear, or change the nature of their stability. If perturbations to $f(y)$ cause changes in the quantitative but not qualitative nature of the critical points, these critical points are called **hyperbolic**.

2. Analysis of Bifurcations

DEFINITION: **bifurcation diagram**

A bifurcation diagram is a picture of the phase lines near a bifurcation value. It appears as a curve in the plane with the autonomous variable y on the vertical axis, and the bifurcation paraemeter on the horizontal axis. Generally a dotted line is used to indicate unstable sections of the curve (i.e. sources) and a solid line is used to indicate stable sections (i.e. sinks).

EXAMPLE Consider the one-parameter family of autonomous DE $\dfrac{dy}{dt} = y^2 + \mu$, where μ is a parameter which can take on any real value. Let's sketch the **bifurcation diagram** of this DE.

This bifurcation is called a **saddle node bifurcation**. This is probably the most typical kind of bifurcation to arise.

THEOREM

Consider a one-parameter family of autonomous DEs where $y' = f(y;\alpha)$ and α is a parameter. The value α_0 will be a **bifurcation value** if and only if $f(y_0;\alpha_0) = 0$ and $f_y(y_0;\alpha_0) = 0$ simultaneously.
(NOTE: This is an 'If and only If" theorem which means the converse is true, i.e. $A \Rightarrow B$ and $B \Rightarrow A$ are both implied. Generally, definitions of quantities are always "If and only If" statements.)

GroupWork

Consider the following three different autonomous ODEs with an unknown real-valued parameter r. Draw bifurcation diagrams for each.
GROUP A: $y' = ry - y^2$
GROUP B: $y' = ry - y^3$
GROUP C: $y' = ry + y^3$

These types of bifurcations are known as the **transcritical**, **supercritical pitchfork** and **subcritical pitchfork** bifurcations, respectively.

Homework

Blanchard, page 107, #8. For the one-parameter family $y' = e^{-y^2} + \alpha$, find the bifurcation values of α and describe the bifurcation that takes place at each value. [HINT: Remember the Linearization Theorem!]