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# Differential Equations

Math 341 Fall 2008  
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MWF 2:30-3:25pm Fowler 307  
<http://faculty.oxy.edu/ron/math/341/08/>

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*Class 6: Wednesday September 10*

**TITLE** Phase Lines and Equilibria

**CURRENT READING** Blanchard, 1.6

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## Homework Assignments due Friday September 12

Section 1.4: 5, 6, 13, 15

Section 1.5: 2, 3, 12, 14, 15.

Section 1.6: 2, 7, 8, 19, 20, 30, 31, 41

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## SUMMARY

We will continue our qualitative analysis of differential equations by learning how to use **phase lines** and the classification of equilibrium points of autonomous ODEs.

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### DEFINITION: critical point

A **critical point** of an autonomous DE  $y' = f(y)$  is a real number  $c$  such that  $f(c) = 0$ .

Another name for critical point is **stationary point** or **equilibrium point**. If  $c$  is a critical point of an autonomous DE, then  $y(x) = c$  is a constant solution of the DE.

### DEFINITION: phase portrait

A **one dimensional phase portrait** of an autonomous DE  $y' = f(y)$  is a diagram which indicates the values of the dependent variable for which  $y$  is increasing, decreasing or constant. Sometimes the vertical version of the phase portrait is called a **phase line**.

### 1. Algorithm For Drawing A Phase Line

- Draw a vertical line
- Find the equilibrium points (i.e. values such that  $f(y) = 0$ ) and mark them on the line
- Find intervals for which  $f(y) > 0$  and mark them with up arrows  $\uparrow$  or  $\wedge$
- Find intervals for which  $f(y) < 0$  and mark them with down arrows  $\downarrow$  or  $\vee$

The textbook likes to have you think of the phase line as a rope with people moving up and down the rope in the directions the arrows are pointing to visualize solutions dynamically.

### EXAMPLE

Consider the autonomous differential equation  $\frac{dy}{dt} = y(a - by)$  where  $a > 0$  and  $b > 0$ .

**1** Find the critical points of the DE.

**2** Determine the values of  $y$  for which  $y(t)$  is increasing and decreasing

**3** Draw the phase line for this DE

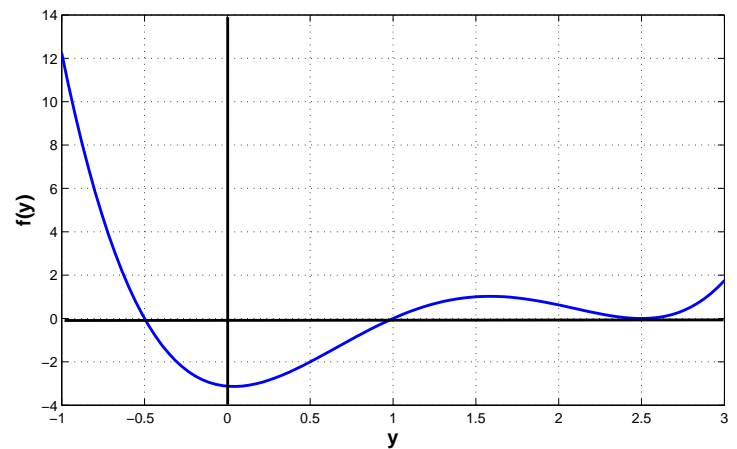
## 2. Obtaining Solution Information from Phase Lines

Consider  $y' = f(y)$  where  $f(y)$  is a continuously differentiable function and  $y(t)$  is a solution to an autonomous ordinary differential equation. The following conclusions can be made

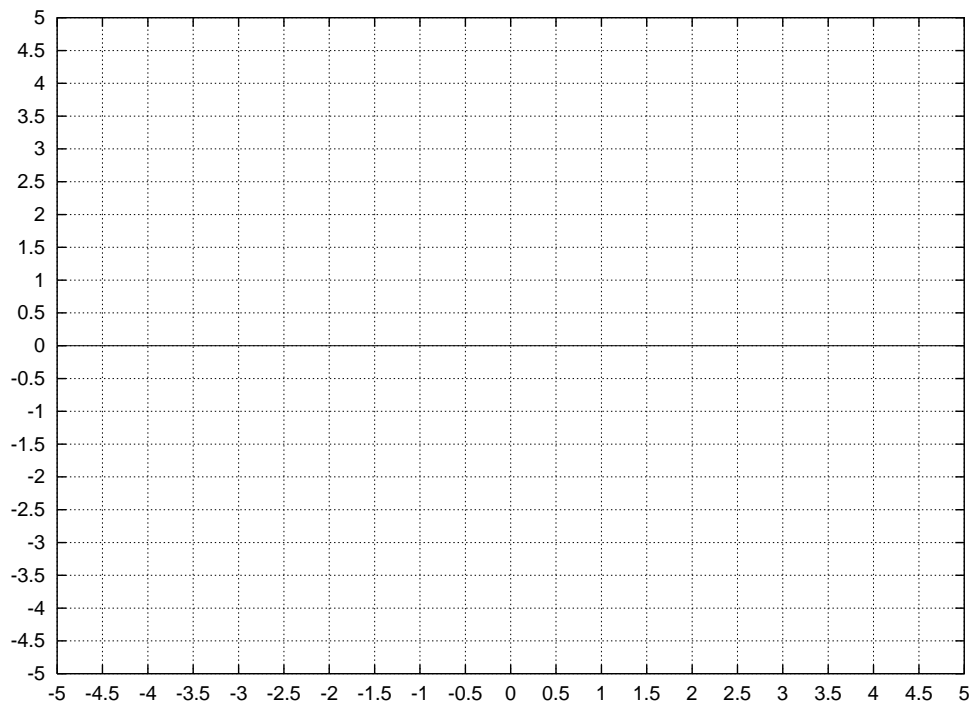
- If  $f(y(0)) = 0$  then  $y(t) = y(0)$  for all  $t$  and  $y(0)$  is an equilibrium point
- If  $f(y(0)) > 0$  then  $y(t)$  is increasing for all  $t$  and either  $y(t) \rightarrow \infty$  as  $t$  increases or  $y(t)$  tends to the first equilibrium point larger than  $y(0)$
- If  $f(y(0)) < 0$  then  $y(t)$  is decreasing for all  $t$  and either  $y(t) \rightarrow -\infty$  as  $t$  increases or  $y(t)$  tends to the first equilibrium point smaller than  $y(0)$

### Exercise

Draw the phase line in the space on the left for the corresponding ODE  $y' = f(y)$  where  $f(y)$  versus  $y$  is graphed below to the right.



Draw graphs of various particular solutions starting at  $y(0) = -1$ ,  $y(0) = 0$ ,  $y(0) = 1$ ,  $y(0) = 2$  and  $y(0) = 3$  in the  $ty$ -plane given below.



### 3. Classifying Equilibrium Points: Sink, Source or Node

A critical value  $c$  is a point where  $y' = f(c) = 0$  splits an interval into two different regions. So there are four possible scenarios for the behavior near  $c$ :  $(+, 0, +)$ ,  $(+, 0, -)$ ,  $(-, 0, +)$  and  $(-, 0, -)$ .

#### **EXAMPLE**

Draw the phase line for each of these cases and then classify the corresponding critical points as asymptotically stable (i.e. attractor or **sink**), unstable (repellor or **source**) or semi-stable **node**.

#### **Exercise**

Draw graphs of the autonomous function  $f(y)$  near equilibrium points classified as a sink, source or node corresponding to the phase line in the example above

**THEOREM: Linearization Theorem**

IF  $y_0$  is an equilibrium point of the autonomous differential equation  $y' = f(y)$  where  $f(y)$  is a continuously differentiable function, THEN

- If  $f'(y_0) < 0$  then  $y_0$  is a sink.
- If  $f'(y_0) > 0$  then  $y_0$  is a source.
- If  $f'(y_0) = 0$  then more information is needed to classify the equilibrium point.

**EXAMPLE**

What can you say about the equilibrium point of the ODE  $y' = y(\cos(y^5 + 2y) - 27\pi y^4)$  at  $y = 0$ ?

**Exercise**

Inspired by **Blanchard, Devaney & Hall, #43, page 95.**

Suppose  $y' = f(y)$  and  $y = y_0$  is an equilibrium point and

- $f'(y_0) = 0$ ,  $f''(y_0) > 0$ : Is  $y_0$  a sink, source or node? EXPLAIN.
- $f'(y_0) = 0$ ,  $f''(y_0) < 0$ : Is  $y_0$  a sink, source or node? EXPLAIN.
- $f'(y_0) = 0$ ,  $f''(y_0) = 0$  and  $f'''(y_0) > 0$ : Is  $y_0$  a sink, source or node? EXPLAIN.
- $f'(y_0) = 0$ ,  $f''(y_0) = 0$  and  $f'''(y_0) < 0$ : Is  $y_0$  a sink, source or node? EXPLAIN.