# Differential Equations

Math 341 Fall 2008 — MWF 2:30-3:25pm Fowler 307

 — `http://faculty.oxy.edu/ron/math/341/08/`

*Class 5*: **Monday September 8**

**TITLE** Existence and Uniqueness of Solutions
**CURRENT READING** Blanchard, 1.5

**Homework Assignments due Friday September 12**
Section 1.4: 5, 6, 13, 15
Section 1.5: 2, 3, 12, 14, 15.
Section 1.6: 2, 7, 8, 19, 20, 30, 31, 41

**SUMMARY**
We will investigate the conditions which guarantee existence and/or uniqueness of solutions to the initial value problem $y' = f(t, y), \quad y(t_0) = y_0$.

## 1. Do Problems Always Have Solutions?

Think about the problem $2x^5 - 10x + 5 = 0$. Does it have a solution? How do we know?

## 2. Existence and Uniqueness Of Particular Solutions

The main questions we would like to be able to answer when analyzing IVPs are:
1) **Existence** Does the differential equation possess solutions which pass through the given initial condition? and
2) **Uniqueness** If such a solution does exist, can we be certain that it is the only one? Luckily, there's a theorem that answers these questions for us.

THEOREM: **Existence of a Unique Solution**

Let $\mathcal{R}$ be a rectangular region in the $xy$-plane defined by $a \le x \le b, c \le y \le d$ that contains the point $(x_0, y_0)$ in its interior. IF $f(x, y)$ and $\partial f/\partial y$ are continuous on $\mathcal{R}$, THEN there exists some interval $I_0$ defined as $x_0 - h < x < x_0 + h$ for $h > 0$ contained in $a \le x \le b$ and a unique function $y(x)$ defined on $I_0$ that is a solution of the initial value problem $y' = f(x, y) \quad y(x_0) = y_0$.

## 3. Theorems Have Hypotheses and Conclusions

The existence and uniqueness theorem is actually two different theorems with different hypotheses and conclusions.
1) **Existence** IF $f(t, y)$ is continuous on a square containing $t_0, y_0$ THEN there exists a solution on an interval $(t_0 - \epsilon, t_0 + \epsilon)$ for some $\epsilon > 0$
2) **Uniqueness** IF $f(t, y)$ and $\dfrac{\partial f}{\partial y}$ are both continuous on a square containing $(t_0, y_0)$ THEN there exists a unique solution

**RECALL**

IF A, THEN B **is equivalent to** (The Contrapositive) IF NOT B, THEN NOT A.
(The Inverse) IF NOT A, THEN NOT B and (The Converse) IF B, THEN A are equivalent to each other, but are NOT equivalent to the original theorem.

For example consider the logical statement: "IF it is raining, THEN the ground is wet."
The contrapositive is "If the ground is NOT wet, then it is NOT raining."
The Converse: IF ______________________ THEN ______________________.
The Inverse: IF ____________________ THEN ________________.

**EXAMPLE**

Show that the initial value problem

$$\frac{dy}{dx} = x\sqrt{y}, \quad y(0) = 0$$

has at least two solutions since the trivial solution $y(x) = 0$ and the solution $y(x) = \frac{1}{16}x^4$ both satisfy the IVP. **Verify this!**

Using the Existence and Uniqueness Theorem, we look at the functions $f(x, y) = x\sqrt{y}$ and $\dfrac{\partial f}{\partial y} = \dfrac{x}{2\sqrt{y}}$. At the origin $(0, 0)$ what can we say about $f(x, y)$ and $f_y(x, y)$?

What can we say about $f(x, y)$ and $f_y(x, y)$ at $(1, 2)$? What does this imply about existence and uniqueness of the corresponding IVP $y' = xy^{1/2}, y(1) = 2$?

### 4. Implications of Existence/Uniqueness Theorem

**Exercise**

Inspired by **Blanchard, Devaney & Hall, #9, page 74**.

(a) Show that $y_1(t) = t^2$ and $y_2(t) = t^2 + 1$ are both solutions of $\dfrac{dy}{dt} = -y^2 + y + 2yt^2 + 2t - t^2 - t^4$

(b) Show that if $y(t)$ is another solution to the given ODE with initial condition $0 < y(0) < 1$ then $t^2 < y(t) < t^2 + 1$ for all $t$

(c) Illustrate your answer by using technology to explore the slope field

**EXAMPLE**

Inspired by **Blanchard, Devaney & Hall, #17**.

Consider the differential equation $\dfrac{dy}{dt} = \dfrac{t}{y-2}, \quad y(-1) = 0$

(a) Find a formula for the solution to the IVP.

(b) State the domain of definition of the solution

(c) Describe what happens as the solution reaches the limit of its domain of definition