

1. We're interested in finding the function $f(t)$ whose Laplace Transform is

$$F(s) = A(s) - B(s) = \frac{1}{s^2} - \frac{e^{-s}}{s(1-e^{-s})}, \quad s > 0$$

(a) 2 points. Compute $\mathcal{L}^{-1} \left[\frac{1}{s^2} \right] = a(t)$.

$$\mathcal{L}^{-1} \left[\frac{1}{s^n} \right] = \frac{t^{n-1}}{(n-1)!}$$

$$a(t) = \frac{t^1}{1!} = t$$

(b) 2 points. If one considers $\frac{1}{1-e^{-s}}$ as the sum of a geometric series $\sum_{k=0}^{\infty} ar^k$ with first term $a = 1$

and ratio $r = e^{-s}$ then show that $\frac{e^{-s}}{s(1-e^{-s})}$ can be written as $\sum_{k=1}^{\infty} \frac{e^{-ks}}{s} = \frac{e^{-s}}{s} + \frac{e^{-2s}}{s} + \frac{e^{-3s}}{s} + \dots$

$$\begin{aligned} \frac{e^{-s}}{s} \frac{1}{1-e^{-s}} &= \frac{e^{-s}}{s} \sum_{k=0}^{\infty} (e^{-s})^k = \frac{1}{s} \sum_{k=0}^{\infty} (e^{-s})^{k+1} = \frac{1}{s} \sum_{k=1}^{\infty} e^{-ks} \\ &= \sum_{k=1}^{\infty} \frac{e^{-ks}}{s} = \frac{e^{-s}}{s} + \frac{e^{-2s}}{s} + \frac{e^{-3s}}{s} + \dots \end{aligned}$$

(c) 3 points. Recall that $\mathcal{L}^{-1}[e^{-as}F(s)] = f(t-a)\mathcal{H}(t-a)$. Using the result given in (b), compute

$$\mathcal{L}^{-1} \left[\frac{e^{-s}}{s(1-e^{-s})} \right] = b(t).$$

$$\mathcal{L}^{-1} \left[\frac{e^{-s}}{s(1-e^{-s})} \right] = \mathcal{L}^{-1} \left[\frac{e^{-s}}{s} \right] + \mathcal{L}^{-1} \left[\frac{e^{-2s}}{s} \right] + \dots$$

$$= \mathcal{H}(t-1) + \mathcal{H}(t-2) + \mathcal{H}(t-3) + \dots = \lfloor t \rfloor = \text{lowest integer function}$$

(d) 3 points. Give a sketch of $a(t)$, $b(t)$ and $f(t) = a(t) - b(t)$ below for $t > 0$ (Use different pairs of axes for each graph.)

