

1. The Gamma Function is defined as

$$\Gamma(\alpha) = \int_0^{\infty} e^{-t} t^{\alpha-1} dt, \quad (\alpha > 0).$$

(a) 1 point. Show that $\Gamma(1) = 1$.

$$\Gamma(1) = \int_0^{\infty} e^{-t} t^{1-1} dt = \lim_{b \rightarrow \infty} \int_0^b e^{-t} dt = \lim_{b \rightarrow \infty} -e^{-t} \Big|_0^b = \lim_{b \rightarrow \infty} -e^{-b} + 1 = 1$$

(b) 2 points. Show that $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$.

$$\Gamma(\alpha+1) = \int_0^{\infty} e^{-t} t^{\alpha} dt = t^{\alpha} e^{-t} \Big|_0^{\infty} - \int_0^{\infty} (-e^{-t}) \alpha t^{\alpha-1} dt$$

$$u = t^{\alpha} \quad du = \alpha t^{\alpha-1}$$

$$dv = e^{-t} \quad v = -e^{-t}$$

$$= \lim_{b \rightarrow \infty} b^{\alpha} e^{-b} - 0 + \alpha \int_0^{\infty} e^{-t} t^{\alpha-1} dt = 0 + \alpha \Gamma(\alpha)$$

$$\Gamma(\alpha+1) = \alpha \Gamma(\alpha)$$

(c) 3 points. Use your results in (a) and (b) to show that $\Gamma(n+1) = n!$, where n is a positive integer. (HINT: use mathematical induction).

Induction: $\Gamma(2) = 2\Gamma(1) = 2 \cdot 1 = 2!$ ($n=1$ case)

$$\Gamma(n) = (n-1)\Gamma(n-1) \Rightarrow \Gamma(n+1) = n!$$

$$\Gamma(n) = (n-1)\Gamma(n-1)$$

$$\Gamma(n+1) = n\Gamma(n) \Rightarrow \Gamma(n+1) = n \cdot (n-1)\Gamma(n-1)$$

$$= n \cdot (n-1)(n-2) \cdots \Gamma(2)$$

$$= n(n-1)(n-2) \cdots 2 \cdot 1$$

$$= n!$$

(d) 4 points. Use all the previous results to show that $\mathcal{L}[t^{\alpha}] = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$ and $\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$ when n is a positive integer.

$$\mathcal{L}[t^{\alpha}] = \int_0^{\infty} e^{-st} t^{\alpha} dt = \int_0^{\infty} e^{-u} \left(\frac{u}{s}\right)^{\alpha} \frac{du}{s} = \frac{1}{s^{\alpha+1}} \int_0^{\infty} e^{-u} u^{\alpha} du = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$$

let $u = st \quad du = s dt$

$$\mathcal{L}[t^n] = \int_0^{\infty} e^{-st} t^n dt = t^n \frac{e^{-st}}{-s} \Big|_0^{\infty} - \int_0^{\infty} \frac{e^{-st}}{-s} n t^{n-1} dt = \frac{n}{s} \int_0^{\infty} t^{n-1} e^{-st} dt$$

$$\mathcal{L}[t^n] = \frac{n}{s} \mathcal{L}[t^{n-1}]$$

$$\mathcal{L}[t^0] = \mathcal{L}[1] = \frac{1}{s}$$

$$\mathcal{L}[t^n] = \frac{n!}{s^n} \cdot \frac{1}{s} = \frac{n!}{s^{n+1}}$$