1. Consider the linear system of ordinary differential equations with a parameter $\alpha$
\[
\frac{dx}{dt} = y + \alpha x(x^2 + y^2) \\
\frac{dy}{dt} = -x + \alpha y(x^2 + y^2)
\]

(a) 2 points. Use linearization to show that regardless of the value of $\alpha$, the equilibrium at $(0,0)$ is a center.

\[J(0,0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \]

\[\text{eig}(J(0,0)) = \lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i \Rightarrow \]

(b) 2 points. Suppose $\alpha \neq 0$. Let $r^2 = x^2 + y^2$. Use the fact that $\frac{dr}{dt} = x \frac{dx}{dt} + y \frac{dy}{dt}$ and the given nonlinear system of ODEs to produce the expression $\frac{dr}{dt} = \alpha r^3$.

\[
\begin{align*}
\frac{x}{dt} &= xy + \alpha x^2(x^2 + y^2) \\
\frac{y}{dt} &= -xy + \alpha y^2(x^2 + y^2)
\end{align*}
\]

(c) 3 points. Suppose $\alpha < 0$ (say $\alpha = -1$). Using separation of variables, integrate the differential equation $\frac{dr}{dt} = \alpha r^3$ to obtain an expression for $r(t)$ and show that as $t \to \infty$, $r \to 0$ which means that the center you found in part (a) is asymptotically stable.

\[
\begin{align*}
\int \frac{dr}{r^3} &= \int dt \\
|\frac{2}{r^2}| &= -t + C \\
|\lim_{t \to \infty} r(t)| &= \lim_{t \to \infty} \sqrt{\frac{2}{t-C}} = 0
\end{align*}
\]

(d) 3 points. Suppose $\alpha > 0$ (say $\alpha = +1$). Using separation of variables, integrate the differential equation $\frac{dr}{dt} = \alpha r^3$ to obtain an expression for $r(t)$ and show that as $t$ increases, $r \to +\infty$ which means that the center you found in part (a) is unstable.

\[
\begin{align*}
\int \frac{dr}{r^3} &= \int dt \\
|\frac{2}{r^2}| &= t + C \\
|\lim_{t \to D^-} \frac{2}{\sqrt{D-t}}| &= +\infty
\end{align*}
\]

\[D-t > 0 \Rightarrow D > t \]

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