1. Consider the linear system of ordinary differential equations with a parameter α

$$\frac{dx}{dt} = y + \alpha x(x^2 + y^2)$$
$$\frac{dy}{dt} = -x + \alpha y(x^2 + y^2)$$

(a) *2 points.* Use linearization to show that regardless of the value of α, the equilibrium at $(0,0)$ is a center.

$$J(0,0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$\text{eig}(J(0,0))$ $\quad p(\lambda) = \lambda^2 + 1 = 0$

$\Rightarrow \lambda = \pm i \quad \Rightarrow$

(b) *2 points.* Suppose $\alpha \neq 0$. Let $r^2 = x^2 + y^2$. Use the fact that $r\frac{dr}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt}$ and the given nonlinear system of ODEs to produce the expression $\frac{dr}{dt} = \alpha r^3$.

$x\frac{dx}{dt} = xy + \alpha x^2(x^2+y^2)$

$y\frac{dy}{dt} = -xy + \alpha y^2(x^2+y^2)$

$x\frac{dx}{dt} + y\frac{dy}{dt} = \alpha(x^2+y^2)^2$

$r\frac{dr}{dt} = \alpha(r^2)^2 = \alpha r^4 \Rightarrow \frac{dr}{dt} = \alpha r^3$

(c) *3 points.* Suppose $\alpha < 0$ (say $\alpha = -1$). Using separation of variables, integrate the differential equation $\frac{dr}{dt} = \alpha r^3$ to obtain an expression for $r(t)$ and show that as $t \to \infty$, $r \to 0$ which means that the center you found in part (a) is *asymptotically* **stable.**

$\alpha = -1$

$\frac{dr}{dt} = -r^3$

$\int \frac{dr}{r^3} = -\int dt$

$-\frac{2}{r^2} = -t + C$

$\frac{2}{r^2} = t - C$

$r^2 = \frac{2}{t-C}$

$r(t) = \sqrt{\frac{2}{t-C}}$ where $t - C > 0$

$\Rightarrow t > C$

$\lim_{t\to\infty} r(t) = \lim_{t\to\infty} \sqrt{\frac{2}{t-C}} = 0$

(d) *3 points.* Suppose $\alpha > 0$ (say $\alpha = +1$). Using separation of variables, integrate the differential equation $\frac{dr}{dt} = \alpha r^3$ to obtain an expression for $r(t)$ and show that as t increases, $r \to +\infty$ which means that the center you found in part (a) is **unstable.**

$\alpha = +1$

$\frac{dr}{dt} = r^3$

$\int \frac{dr}{r^3} = \int dt$

$-\frac{2}{r^2} = t + C$

$\frac{2}{r^2} = -t - C$

$r^2 = \frac{2}{-t-C}$

let $D = -C$

$r^2 = \frac{2}{-t+D}$

$r^2 = \frac{2}{D-t}$

$r = \sqrt{\frac{2}{D-t}}$

$D - t > 0 \Rightarrow D > t$

$\lim_{t\to D^-} \sqrt{\frac{2}{D-t}} = +\infty$