

1. Consider the system of ordinary differential equations

$$\frac{d\vec{x}}{dt} = A\vec{x} = \begin{bmatrix} 0 & 2 \\ 0 & -1 \end{bmatrix} \vec{x} \text{ where } \vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

(a) 4 points. Show that the matrix  $A$  has eigenvalues 0 and  $-1$  and eigenvectors which are multiples of  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$  and  $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ . Write down the general solution of the system.

$$\begin{pmatrix} 0 & 2 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\vec{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \lambda = 0$$

$$\begin{pmatrix} 0 & 2 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} = -1 \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$$\vec{x} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad \lambda = -1$$

$$\vec{x} = c_1 e^{-t} \begin{pmatrix} -2 \\ 1 \end{pmatrix} + c_2 e^{0t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = c_1 e^{-t} \begin{pmatrix} -2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

(b) 3 points. Find the exact solution for each of the trajectories which go through the points  $(1, 1)$ ,  $(0, -2)$  and  $(4, 0)$ .

$$t=0, \begin{pmatrix} 1 \\ 1 \end{pmatrix} = c_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow c_1 = 1, c_2 = 3$$

$$\vec{x}(t) = \begin{pmatrix} -2e^{-t} + 3 \\ e^{-t} \end{pmatrix}$$

$$t=4, \begin{pmatrix} 0 \\ -2 \end{pmatrix} = c_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad c_1 = -2, c_2 - 2c_1 = 0 \Rightarrow c_2 = 2c_1 = -4$$

$$\vec{x}(t) = -2e^{-t} \begin{pmatrix} -2 \\ 1 \end{pmatrix} - 4 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4e^{-t} - 4 \\ -2e^{-t} \end{pmatrix}$$

$$t=0, \begin{pmatrix} 4 \\ 0 \end{pmatrix} = c_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow c_1 = 0, c_2 = 4$$

$$\vec{x}(t) = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$$

(c) 3 points. On the figure below clearly indicate where each of the trajectories of the solutions which start at  $(1, 1)$ ,  $(0, -2)$  and  $(4, 0)$  ends up as  $t \rightarrow \infty$ .

