2. The general solution to the associated homogeneous equation is $y_h(t) = ke^{-4t}$. For a particular solution of the nonhomogeneous equation, we guess a solution of the form $y_p(t) = \alpha e^{-t}$. Then

$$\begin{aligned}\frac{dy_p}{dt} + 4y_p &= -\alpha e^{-t} + 4\alpha e^{-t} \\ &= 3\alpha e^{-t}.\end{aligned}$$

Consequently, we must have $3\alpha = 3$ for $y_p(t)$ to be a solution. Hence, $\alpha = 1$, and the general solution to the nonhomogeneous equation is

$$y(t) = ke^{-4t} + e^{-t}.$$

3. The general solution to the associated homogeneous equation is $y_h(t) = ke^t$. For a particular solution of the nonhomogeneous equation, we guess a solution of the form $y_p(t) = \alpha \cos 2t + \beta \sin 2t$. Then

$$\begin{aligned}\frac{dy_p}{dt} - y_p &= -2\alpha \sin 2t + 2\beta \cos 2t - (\alpha \cos 2t + \beta \sin 2t) \\ &= (2\beta - \alpha) \cos 2t + (-2\alpha - \beta) \sin 2t\end{aligned}$$

Consequently, we must have

$$(2\beta - \alpha) \cos 2t + (-2\alpha - \beta) \sin 2t = \cos 2t$$

for $y_p(t)$ to be a solution. We must solve

$$\begin{cases} 2\beta - \alpha = 1 \\ -2\alpha - \beta = 0. \end{cases}$$

Hence, $\alpha = -1/5$ and $\beta = 2/5$. The general solution is

$$y(t) = ke^t - \tfrac{1}{5} \cos 2t + \tfrac{2}{5} \sin 2t.$$

4. The general solution to the associated homogeneous equation is $y_h(t) = ke^{2t}$. For a particular solution of the nonhomogeneous equation, we guess $y_p(t) = \alpha \cos 2t + \beta \sin 2t$. Then

$$\begin{aligned}\frac{dy_p}{dt} - 2y_p &= -2\alpha \sin 2t + 2\beta \cos 2t - 2(\alpha \cos 2t + \beta \sin 2t) \\ &= (2\beta - 2\alpha) \cos 2t + (-2\alpha - 2\beta) \sin 2t.\end{aligned}$$

Consequently, we must have

$$(2\beta - 2\alpha) \cos 2t + (-2\alpha - 2\beta) \sin 2t = \sin 2t$$

for $y_p(t)$ to be a solution, that is, we must solve

$$\begin{cases} -2\alpha - 2\beta = 1 \\ -2\alpha + 2\beta = 0. \end{cases}$$

Hence, $\alpha = -1/4$ and $\beta = -1/4$. The general solution of the nonhomogeneous equation is

$$y(t) = ke^{2t} - \tfrac{1}{4}\cos 2t - \tfrac{1}{4}\sin 2t.$$

5. The general solution to the associated homogeneous equation is $y_h(t) = ke^{4t}$. For a particular solution of the nonhomogeneous equation, we guess $y_p(t) = \alpha te^{4t}$ rather than αe^{4t} because αe^{4t} is a solution of the homogeneous equation. Then

$$\begin{aligned}\frac{dy_p}{dt} - 4y_p &= \alpha e^{4t} + 4\alpha te^{4t} - 4\alpha te^{4t} \\ &= \alpha e^{4t}.\end{aligned}$$

Consequently, we must have $\alpha = -5$ for $y_p(t)$ to be a solution. Hence, the general solution to the nonhomogeneous equation is

$$y(t) = ke^{4t} - 5te^{4t}.$$

6. The general solution of the associated homogeneous equation is $y_h(t) = ke^{t/2}$. For a particular solution of the nonhomogeneous equation, we guess $y_p(t) = \alpha te^{t/2}$ rather than $\alpha e^{t/2}$ because $\alpha e^{t/2}$ is a solution of the homogeneous equation. Then

$$\begin{aligned}\frac{dy_p}{dt} - \frac{y_p}{2} &= \alpha e^{t/2} + \frac{\alpha}{2}te^{t/2} - \frac{\alpha te^{t/2}}{2} \\ &= \alpha e^{t/2}.\end{aligned}$$

Consequently, we must have $\alpha = 4$ for $y_p(t)$ to be a solution. Hence, the general solution to the nonhomogeneous equation is

$$y(t) = ke^{t/2} + 4te^{t/2}.$$

7. The general solution to the associated homogeneous equation is $y_h(t) = ke^{-2t}$. For a particular solution of the nonhomogeneous equation, we guess a solution of the form $y_p(t) = \alpha e^{t/3}$. Then

$$\begin{aligned}\frac{dy_p}{dt} + 2y_p &= \tfrac{1}{3}\alpha e^{t/3} + 2\alpha e^{t/3} \\ &= \tfrac{7}{3}\alpha e^{t/3}.\end{aligned}$$

Consequently, we must have $\frac{7}{3}\alpha = 1$ for $y_p(t)$ to be a solution. Hence, $\alpha = 3/7$, and the general solution to the nonhomogeneous equation is

$$y(t) = ke^{-2t} + \tfrac{3}{7}e^{t/3}.$$

Since $y(0) = 1$, we have

$$1 = k + \tfrac{3}{7},$$

so $k = 4/7$. The function $y(t) = \frac{4}{7}e^{-2t} + \frac{3}{7}e^{t/3}$ is the solution of the initial-value problem.

8. The general solution to the associated homogeneous equation is $y_h(t) = ke^{2t}$. For a particular solution of the nonhomogeneous equation, we guess a solution of the form $y_p(t) = \alpha e^{-2t}$. Then

$$\begin{aligned}\frac{dy_p}{dt} - 2y_p &= -2\alpha e^{-2t} - 2\alpha e^{-2t} \\ &= -4\alpha e^{-2t}.\end{aligned}$$

Consequently, we must have $-4\alpha = 3$ for $y_p(t)$ to be a solution. Hence, $\alpha = -3/4$, and the general solution to the nonhomogeneous equation is

$$y(t) = ke^{2t} - \tfrac{3}{4}e^{-2t}.$$

Since $y(0) = 10$, we have

$$10 = k - \tfrac{3}{4},$$

so $k = 43/4$. The function

$$y(t) = \tfrac{43}{4}e^{2t} - \tfrac{3}{4}e^{-2t}$$

is the solution of the initial-value problem.

9. The general solution of the associated homogeneous equation is $y_h(t) = ke^{-t}$. For a particular solution of the nonhomogeneous equation, we guess a solution of the form $y_p(t) = \alpha \cos 2t + \beta \sin 2t$. Then

$$\begin{aligned}\frac{dy_p}{dt} + y_p &= -2\alpha \sin 2t + 2\beta \cos 2t + \alpha \cos 2t + \beta \sin 2t \\ &= (\alpha + 2\beta)\cos 2t + (-2\alpha + \beta)\sin 2t.\end{aligned}$$

Consequently, we must have

$$(\alpha + 2\beta)\cos 2t + (-2\alpha + \beta)\sin 2t = \cos 2t$$

for $y_p(t)$ to be a solution. We must solve

$$\begin{cases} \alpha + 2\beta = 1 \\ -2\alpha + \beta = 0. \end{cases}$$

Hence, $\alpha = 1/5$ and $\beta = 2/5$. The general solution to the differential equation is

$$y(t) = ke^{-t} + \tfrac{1}{5}\cos 2t + \tfrac{2}{5}\sin 2t.$$

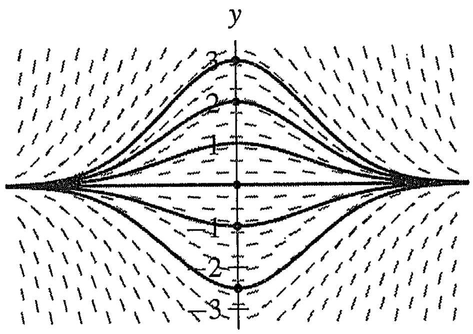
To find the solution of the given initial-value problem, we evaluate the general solution at $t = 0$ and obtain

$$y(0) = k + \tfrac{1}{5}.$$

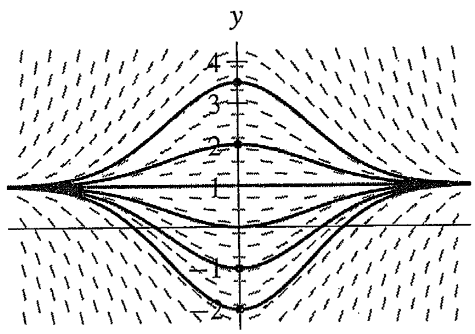
Since the initial condition is $y(0) = 5$, we see that $k = 24/5$. The desired solution is

$$y(t) = \tfrac{24}{5}e^{-t} + \tfrac{1}{5}\cos 2t + \tfrac{2}{5}\sin 2t.$$

15. The Linearity Principle says that all nonzero solutions of a homogeneous linear equation are constant multiples of each other.



16. The Extended Linearity Principle says that any two solutions of a nonhomogeneous linear equation differ by a solution of the associated homogeneous equation.



17. **(a)** We compute

$$\frac{dy_1}{dt} = \frac{1}{(1-t)^2} = (y_1(t))^2$$

to see that $y_1(t)$ is a solution.

(b) We compute

$$\frac{dy_2}{dt} = 2\frac{1}{(1-t)^2} \neq (y_2(t))^2$$

to see that $y_2(t)$ is not a solution.

(c) The equation $dy/dt = y^2$ is not linear. It contains y^2.

18. **(a)** The constant function $y(t) = 2$ for all t is an equilibrium solution.

(b) If $y(t) = 2 - e^{-t}$, then $dy/dt = e^{-t}$. Also, $-y(t) + 2 = e^{-t}$. Consequently, $y(t) = 2 - e^{-t}$ is a solution.

(c) Note that the solution $y(t) = 2 - e^{-t}$ has initial condition $y(0) = 1$. If the Linearity Principle held for this equation, then we could multiply the equilibrium solution $y(t) = 2$ by $1/2$ and obtain another solution that satisfies the initial condition $y(0) = 1$. Two solutions that satisfy the same initial condition would violate the Uniqueness Theorem.

19. Let $y(t) = y_h(t) + y_1(t) + y_2(t)$. Then

$$\begin{aligned}\frac{dy}{dt} + a(t)y &= \frac{dy_h}{dt} + \frac{dy_1}{dt} + \frac{dy_2}{dt} + a(t)y_h + a(t)y_1 + a(t)y_2 \\ &= \frac{dy_h}{dt} + a(t)y_h + \frac{dy_1}{dt} + a(t)y_1 + \frac{dy_2}{dt} + a(t)y_2 \\ &= 0 + b_1(t) + b_2(t).\end{aligned}$$

This computation shows that $y_h(t) + y_1(t) + y_2(t)$ is a solution of the original differential equation.

20. If $y_p(t) = at^2 + bt + c$, then

$$\begin{aligned}\frac{dy_p}{dt} + 2y_p &= 2at + b + 2at^2 + 2bt + 2c \\ &= 2at^2 + (2a + 2b)t + (b + 2c).\end{aligned}$$

Then $y_p(t)$ is a solution if this quadratic is equal to $3t^2 + 2t - 1$. In other words, $y_p(t)$ is a solution if

$$\begin{cases} 2a = 3 \\ 2a + 2b = 2 \\ b + 2c = -1. \end{cases}$$

From the first equation, we have $a = 3/2$. Then from the second equation, we have $b = -1/2$. Finally, from the third equation, we have $c = -1/4$. The function

$$y_p(t) = \tfrac{3}{2}t^2 - \tfrac{1}{2}t - \tfrac{1}{4}$$

is a solution of the differential equation.

21. To find the general solution, we use the technique suggested in Exercise 19. We calculate two particular solutions—one for the right-hand side $t^2 + 2t + 1$ and one for the right-hand side e^{4t}.

With the right-hand side $t^2 + 2t + 1$, we guess a solution of the form

$$y_{p_1}(t) = at^2 + bt + c.$$

Then

$$\begin{aligned}\frac{dy_{p_1}}{dt} + 2y_{p_1} &= 2at + b + 2(at^2 + bt + c) \\ &= 2at^2 + (2a + 2b)t + (b + 2c).\end{aligned}$$

Then y_{p_1} is a solution if

$$\begin{cases} 2a = 1 \\ 2a + 2b = 2 \\ b + 2c = 1. \end{cases}$$