Report on Test 2 Prof. Ron Buckmire

Point Distribution (N=13)

Range	100+	92+	90+	85+	80+	77+	73+	68+	65+	60+	55+	50+	50-
Grade	A+	A	A-	B+	B	B-	C+	C	C-	D+	D	D-	F
Frequency	0	2	0	4	1	1	0	2	0	0	1	2	0

Summary The exam was designed to review the most important concepts in Chapters 3 and 5: Linear and Nonlinear Systems of Ordinary Differential Equations. This exam was more oriented towards calculations-based learners but there were questions designed for students with more visual and verbal learning styles as well. Overall, class performance was encouraging, somewhat improved from Test 1. The mean score was 76. The median score was 81. The high score was 98.

#1 Linear Systems of Differential Equations. This problem is about applying the general idea of being able to solve $\frac{d\vec{x}}{dt} = A\vec{x}$. **(a)** This is a warm-up problem, which basically tests do you understand the notation. Moving from matrix form to a system produces: $\dot{x} = y + z$, $\dot{y} = x + z$ and $\dot{z} = x + y$. **(b)**. Of course one can find eigenvalues by finding the roots of the characteristic polynomial $p(\lambda) = \det(A - \lambda\mathcal{I})$. But when A is a 3×3 like in this problem the polynomial will be a cubic, which could be tricky to solve. So, instead I gave you the eigenvectors and relied on your understanding of the relationship between eigenvalues and eigenvectors, i.e. $A\vec{x} = \lambda\vec{x}$ to fid the corresponding eigenvalues. It turns out that $E_2 = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}$ while $E_{-1} = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}\right\}$. Oh no, a repeated eigenvalue! But there's no reason to panic because the eigenvalue $\lambda = -1$ also happens to have geometric multiplicity equal to its algebraic multiplicity. In other words, it has two eigenvectors. **(c)** Thus the general solution is $\vec{x} = c_1 e^{2t}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_3 e^{-t}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$. **(d)**. To check your solution in (c) is correct you need to check that it the LEFT and RIGHT sides of the equation $\frac{d\vec{x}}{dt} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\vec{x}$ are equal. In other words, differentiation on one hand and matrix multipication on the other.

#2 Equilibria of Planar Systems, Hamiltonian, Trace-Determinant Bifurcation. This question is all about $\frac{d\vec{x}}{dt} = \begin{bmatrix} \alpha & 4 \\ 1 & 1 \end{bmatrix}\vec{x}$. bf (a) "A Hamiltonian function $H(x, y)$ for the given system of ODEs exists only when $\alpha = -1$." **TRUE.** Recall that a planar system $\dot{x} = f(x, y)$, $\quad \dot{y} = g(x, y)$ has Hamiltonian function if $f_x = -g_y$. So considering your given system is $\dot{x} = \alpha x + 4y$ and $\dot{y} = x + y$ that would mean that $f_x = -g_y$ when $\alpha = -1$. **(b)** "The curve in the Trace-Determinant plane corresponding to the matrices for all possible values of α is a line through the origin." **FALSE.** Well, the trace $T = \alpha + 1$ and the determinant $D = \alpha - 4$ so how do they change when α changes? It depends on how they depend on each other! $T - 1 = \alpha = D + 4$ so that $D = T - 5$. In other words, as α changes it traces out the curve $D = T - 5$ in the TD-plane, which repsents a line, that does NOT go through the origin. **(c)** "There is no value of α for which the phase portrait of $\frac{d\vec{x}}{dt} = \begin{bmatrix} \alpha & 4 \\ 1 & 1 \end{bmatrix}\vec{x}$ near the origin will look like the given figure." **TRUE.** Clearly from (b) the curve in the Trace-Determinant plane hits the $T = 0$ (D-axis) at a value less than zero, so it is NOT in the region where centers occur. One could also check that the eigenvalues of A that depend on α have no value of α whereby they are complex with zero real part.

#3 Non-linear Systems of ODEs, Linearization. The nonlinear system is $\dot{x} = -\alpha - x + y, \quad \dot{y} = -4x + y + x^2$. **a)** In his case it turns out it is easie to solve the system to find the equilibria. $y = \alpha + x$ and $0 = -4x + \alpha + x) + x^2$ leads to a simple quadratic $x^2 - 3x + \alpha = 0$ which when solves using the quadratic formula yields the result. **(b)** When $\alpha = 0$ one can plug the value into the formulas give in part (a) to obtai the two equilibrium points $(3, 3)$ and $(0, 0)$. In order to classify the equilibria one needs to obtain the Jacobian, which turns out to be $J(x, y) = \begin{bmatrix} -1 & 1 \\ 2x - 4 & 1 \end{bmatrix}$ so hat $J(0, 0) = \begin{bmatrix} -1 & 1 \\ -4 & 1 \end{bmatrix}$ and $J(3, 3) = \begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix}$. The latter has eigenvalues $\pm\sqrt{3}$ while the former has eigenvalue $\pm\sqrt{-3}$ which means the first is a saddle near (3,3) and a center near the origin. **(c)** Since the equilibria occur at $\left(\frac{3 + \sqrt{9 - 4\alpha}}{2}, \frac{3 + 2\alpha + \sqrt{9 - 4\alpha}}{2}\right)$ and $\left(\frac{3 - \sqrt{9 - 4\alpha}}{2}, \frac{3 + 2\alpha - \sqrt{9 - 4\alpha}}{2}\right)$ it is clear that when $\sqrt{9 - 4\alpha} = 0$ there will only be one equilibrium point and the solution will bifurcate. Thus $\alpha_B = \frac{9}{4}$. **(d)** When $\alpha = \frac{9}{4}$, $x = \frac{3}{2}$ and $y = \frac{15}{4}$. $J(\frac{3}{2}, \frac{15}{4}) = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$ which has Trace equal to zero and determinant equal to zero! So $\lambda = 0, 0$. This is a repeated zero eigenvalue. It turns out that this would mean that the phase portrait near this equlibrium point should have linear solutions parallel to the eigenvector for $\lambda = 0$ which by inspection one can see is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ but that this behavior will probably only be valid very close to the equilibrium point.