Report on Test 1

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Point Distribution (N=13)

Range	100+	90+	87+	85+	76+	73+	70+	64+	60+	50+	40+	30+	30-
Grade	A+	A	A-	B+	B	B-	C+	C	C-	D+	D	D-	F
Frequency	0	2	0	0	4	0	1	2	1	0	1	1	1

Summary The exam was designed to review the most important concepts in the course so far: Existence and Uniqueness theorem, Bifurcation and the ability to find equilibria and sketch solutions of autonomous ODEs, Euler's Method, and the Linearity Principle. In addition, the questions were designed so that were equally weighted towards students with different kinds of learning skills: verbal learners, visual learners and calculations-based learners. However, overall, class performance was somewhat disappointing. The mean score was 66. The median score was 77. The high score was 91.

#1 **Existence and Uniqueness.** This problem is about your understanding of how theorems work in general, and of the Existence and Uniqueness Theorem, in particular. This problem is oriented towards verbal learners and calculations-based learners. **(a)** The EUT states that given an IVP $y' = f(t,y), \quad y(t_0) = y_0$ IF $f(t,y)$ is continuous in a small region containing (t_0, y_0) THEN **there exists** a solution $y(t)$ along a finite interval containing t_0 AND IF $\dfrac{\partial f}{\partial y}$ is continuous in a similar region containing (t_0, y_0) then the solution $y(t)$ will be **unique** for a finite interval containing t_0. So, one needs to think of a function $f(t,y)$ that is not continuous at some point but is simple enough that one can solve the ODE (using separation of variables). The one I came up with was $y' = \sqrt{y}$, $y(0) = 0$. This function $f(t,y) = \sqrt{y}$ is not continuous at $(0,0)$ because one can't take limits from the left, but more importantly, $f_y(t,y) = \dfrac{1}{2\sqrt{y}}$ which is also not continuous at $y = 0$ because of division by zero. (Also, note the notation here. This is NOT $f'(t,y)$ or y'', it is the partial derivative of f with respect to y.) My example violates both the existence *and* uniqueness hypotheses. What can we conclude? NOTHING! When a theorem's hypothesis is FALSE, no conclusion can be made from the theorem. It turns out that at least one solution exists to my IVP, i.e. the equilibrium solution $y(t) = 0$. However, solving the IVP using separation of variables produces another (non-unique!) solution $y(t) = \dfrac{t^2}{4}$.

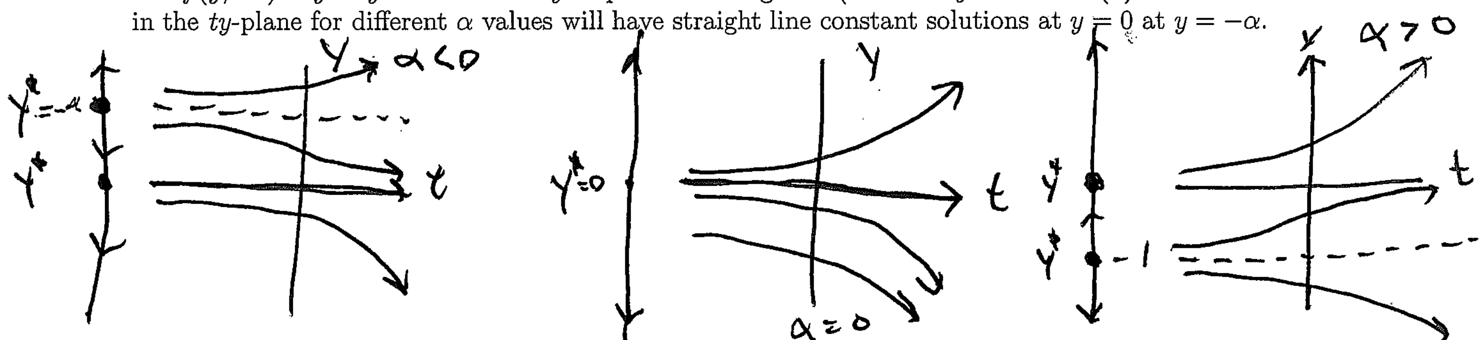
$$\begin{aligned}
\frac{dy}{dt} &= \sqrt{y} \\
\frac{dy}{\sqrt{y}} &= dt \\
\int \frac{dy}{\sqrt{y}} &= \int dt \\
2\sqrt{y} &= t + C \\
2\sqrt{0} &= 0 + C \qquad \text{Using the initial condition } y(0) = 0 \\
0 &= C \\
2\sqrt{y} &= t \\
4y &= t^2 \\
y &= \frac{t^2}{4}
\end{aligned}$$

(b) Coming up with an example IVP that satisfies the uniqueness hypothesis is a repeat of Question 1 from Reading Quiz 2. The simplest choice is $y' = y$, $y(0) = 1$. Clearly $f(t,y) = y$ and $f_y(t,y) = 1$ are both polynomial functions so they are continuous everywhere. So, by the EUT, the solution to my IVP will be unique for any initial condition. That solution is $y(t) = e^t$ and this is the ONLY solution to this IVP. We can make this conclusion because the hypothesis of the existence and uniqueness theorem have been satisfied, so its conclusions are true also.

#2 **Mathematical Modeling, Euler's Method.** This question is actually from Calculus 1. It assesses your ability to convert a sentence into an initial value problem as well as your understanding of how Euler's Method works. This problem is oriented towards "calculations-based" learners. **(a)** Since the rate of change of Tribble population P is proportional to the number of Tribbles, this means that $\dfrac{dP}{dt} \propto P^2$ which implies that $\frac{dP}{dt} = kP^2$. Initially, there are two Tribbles so $P(0) = 2$. The only reason the sentence about the growth rate of Tribbles being 1 when $P = 10$ is to be able to get a value for the proportionality constant k. Plugging in the values the differential equation becomes $1 = k10^2$ so that $k = 0.01$. **(b)** If you need to estimate $P(t)$ two days from now and you know $P(\text{now})$ and $P'(\text{now})$ there's no reason not to use Euler's Method with the largest possible Δt, i.e. $\Delta t = 2$. In other words, $P(\text{then}) \approx P(\text{now}) + P'(\text{now})\Delta t$. **(c)** To determine whether Euler's Method will be an over-estimate or under-estimate you need to know the sign of $P''(\text{now})$. Since $P' = 0.01P^2$, $P'' = 0.01 * 2P * P' = 0.02P * 0.01 * P^2$ which is clearly always positive. Therefore, Euler's Method will produce an over-estimate of the solution at a later time because the unknown solution $P(t)$ is concave up (and increasing at a faster rate than Euler's Method expects).

#3 Systems of ODEs, Analytical Techniques, Linearity Principle. These are TRUE or FALSE questions. They generally assess your ability to explain your reasoning of particularly important ideas. The chosen problems are oriented towards verbal and calculations-based learners. **(a)** "The initial value problem $y' + 2y = 3, y(0) = 1$ possesses the one-parameter family of solutions $y(x) = \frac{1}{2}(3 - Ce^{-2x})$." **FALSE.** It is true that with no initial condition, $y(x) = \frac{1}{2}(3 - Ce^{-2x})$ is a family of solutions, but since you DO have an initial condition, that gives you a choice for C, which turns out to be $C = 1$. Thus there is only one specific solution to this IVP, as the existence and uniqueness theorem would tell you. The point of this question is understanding that solutions of IVPs are particular, solitary functions in most cases, not families of functions. **(b)** "For the system $x' = -3x$, $y' = -2y$ a solution starting at (a, b) when $t = 0$ will approach the origin as $t \to \infty$ for all non-zero real values of a and b." **TRUE.** This is a decoupled linear system so solve it! $x' = -3x, \quad x(0) = a$. Clearly, $x(t) = ae^{-3t}$. Also, $y' = -2y, \quad y(0) = b$ so $y(t) = be^{-2t}$. What happens as t grows without bound? $x(t)$ and $y(t)$ will approach zero, very very quickly. Another way to think of this problem is to consider the direction field and to realize that every arrow you draw will be towards the origin, so any solution starting away from (0,0) will end up there eventually. But one needs to write that sentence down, not assume that a picture will do the talking for you. **(c)** "The linear ordinary differential equation $\frac{dy}{dt} = 2 - y$ has solutions of the form $y_1(t) + y_2(t)$ where $y_1(t) = 2$ and $y_2(t)$ is *any* real number multiple of e^{-t}" **TRUE.** The linearity principle says that the general solution $y_g(t)$ to a non-homogeneous linear ODE is the sum of the homogeneous solution $y_h(t)$ and the non-homogeneous solution $y_p(t)$. The given ODE $y' = 2 - y$ clearly has $y = 2$ as a solution, and this is a solution to a non-homogeneous ODE. The corresponding homogeneous DE is $y' + y = 0$ and clearly has $y = Ae^{-t}$ as a solution where A is any number. Thus the general solution to $y' = 2 - y$ is $2 + Ae^{-t}$.

#4 Bifurcation. This problem is about going through the process of illustrating your understanding of all the aspects of bifurcation on the equilibrium solutions of an autonomous differential equation **without** having to solve it. This problem is oreinted towards visual learners. **(a)** $f(y) = y^3 + \alpha y^2$ is equal to zero when $y = 0$ or $y = -\alpha$ since $y^3 + \alpha y^2 = (y + \alpha)y^2 = 0$. Thus $y^* = 0$ and $y^* = -\alpha$. **(b)** If there is a bifurcation value, it occurs at a value of α for which $f(y; \alpha) = 0$ AND $f_y(y; \alpha) = 0$ simultaneously. $f_y(y; \alpha) = 3y^2 + 2y\alpha$. Clearly, when $y = 0$ both equations are satisfied, but what about at $y = -\alpha$? The first is satisfied,and the second becomes $3(-\alpha)^2 + 2(-\alpha)\alpha = \alpha^2 = 0$ which implies $\alpha = 0$. So, the coordinates of the bifurcation point in the αy-plane are $(0, 0)$ with $\alpha_B = 0$. The easiest way to draw the phase lines for different α values is pick a value above α_B (like 1) and one below it (like -1) and graph the corresponding functions $f(y; 1) = y^3 + y^2$ and $f(y; -1) = y^3 - y^2$ to see where y' is positive or negative (and thus y is $\wedge$ or $\vee$. **(c)** The solution curves in the ty-plane for different α values will have straight line constant solutions at $y = 0$ at $y = -\alpha$.



(d) The bifurcation diagram shows that along $y = 0$ every point is a **node**, while along the line $y = -\alpha$ every point is a **source** with a bifurcation value at $(0, 0)$.