Math 341–Fall 2006

Midterm Exam 1–October 9th, 2006

Name: BUCKMIRE

Only a calculator and some writing instrument are allowed. You cannot use any differential equation solving or integrating capacities that are on your calculator. You can also use your "personalized" cheat sheet, simply attach it to the exam at the end.

1. *(7.5 points)* Consider the initial value problem

$$\frac{dy}{dx} = 2xy^2, \quad y(0) = \frac{1}{c^2}.$$

For what values of c do unique solutions to the IVP exist? Explain your answer.

$f(x,y) = 2xy^2$

$f_y(x,y) = 4xy$

By E&U Theorem $y' = 2xy^2$, $y(0) = \frac{1}{c^2}$ will have unique sol'ns for all IVP $c \neq 0$.

2. *(7.5 points)* Consider the following second order differential equation:

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} + 4x = 0.$$

True/False? The differential equation can be written as a linear system of first-order equations with only $(0,0)$ as an equilibrium value. If false, explain why. If true, write the system in matrix form and indicate how you know about its equilibria.

$y = x'$

$y' = x'' = -x' - 4x$

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} y \\ -y-4x \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

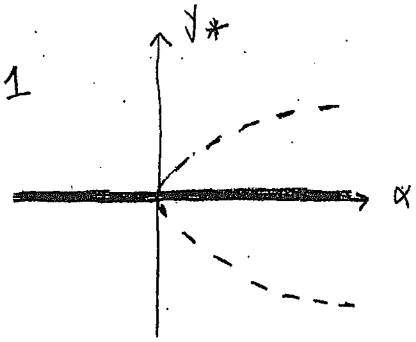
$\vec{x}' = A\vec{x}$ thus $\vec{x} = \vec{0}$ is an equilibrium value.

3. *(15 points)* Here are two families of differential equations:

$$\text{A.}\ \frac{dy}{dt} = \alpha y^2 - y^4, \quad \text{B.}\ \frac{dy}{dt} = \alpha y - y^3.$$

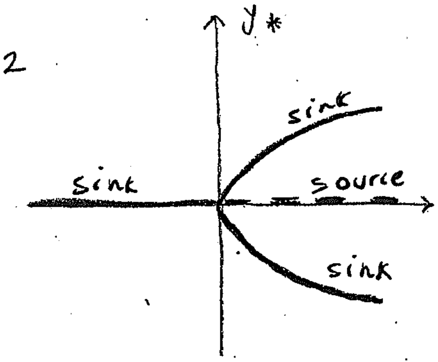
Below are four bifurcation diagrams. Below the figures, match the number of the bifurcation diagram with the letter of the differential equation. If no bifurcation diagram corresponds to a given equation, write "none" next to that letter. Explain in 1-2 sentences why your choices are correct.

\#1



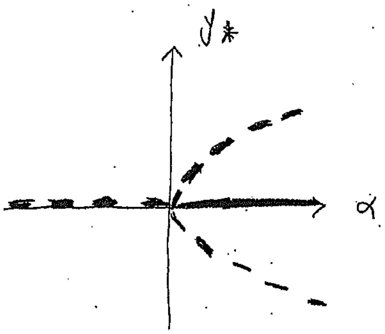
NONE

\# 2 B.



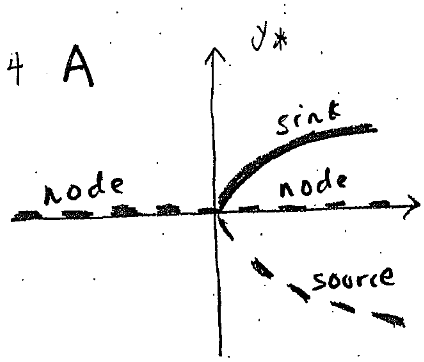
This is the typical pitchfork bifurcation

\#3



NONE

\#4 A



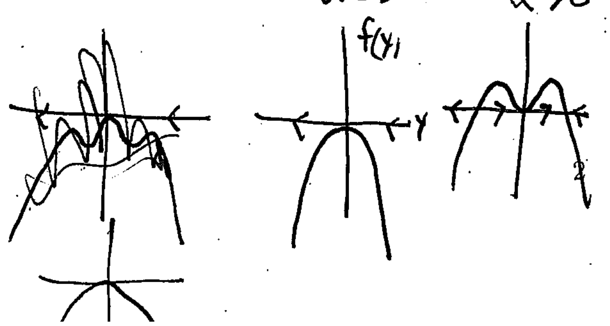
If one considers - - - as unstable, then the

A. $y' = (\alpha - y^2)y^2 = 0$

$y = 0,\ y^2 = \alpha$

$y = \pm\sqrt{\alpha}$

$\alpha < 0$ $\quad$ $\alpha = 0$ $\quad$ $\alpha > 0$

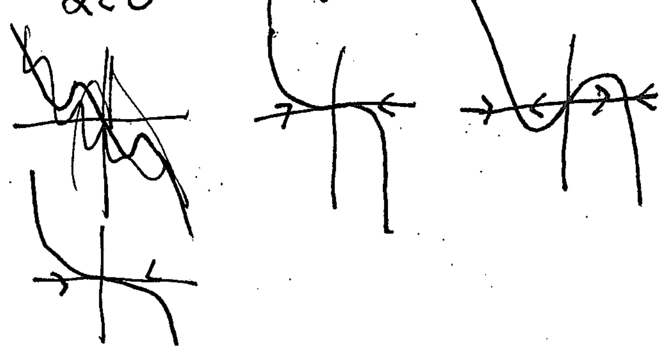


B. $y' = (\alpha - y^2)y = 0$

$y = 0,\ y^2 = \alpha$

$y = \pm\sqrt{\alpha}$

$\alpha < 0$ $\quad$ $\alpha = 0$ $\quad$ $\alpha > 0$



4. *(20 points)* Determine if the following differential equations has a source at $y = 1$. Explain why or why not in a sentence in each case.

A. $\frac{dy}{dt} = 1 - \sqrt{y}$

$f(y) = 1 - \sqrt{y} = 0 \Rightarrow y = 1$

$f'(y) = -\frac{1}{2\sqrt{y}}$

$f'(1) = -\frac{1}{2} \Rightarrow$ ~~source~~ sink at $y = 1$

B. $\frac{dy}{dt} = y^2 + 4y$

$y^2 + 4y = 0$

$y = 0$ is an equilibrium, so $y=1$ is not a source.

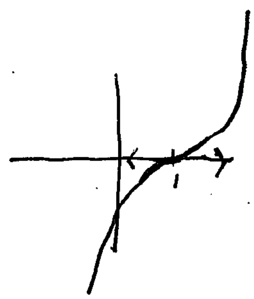
C. $\frac{dy}{dt} = (y-1)^3$

$f(y) = (y-1)^3 = 0$

$\Rightarrow y = 1$

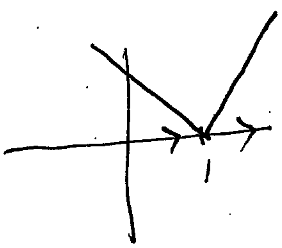
$f'(y) = 3(y-1)^2 \cdot 1$

$f'(1) = 0$



There's a source at $y=1$ from the graph

D. $\frac{dy}{dt} = |1 - y|$



$y=1$ is an equilibrium, but it is a node.

5. *(15 points)* For the following differential equation:

$$\frac{dy}{dt} = y^2 - 2yt + t^2 + y - t + 1,$$

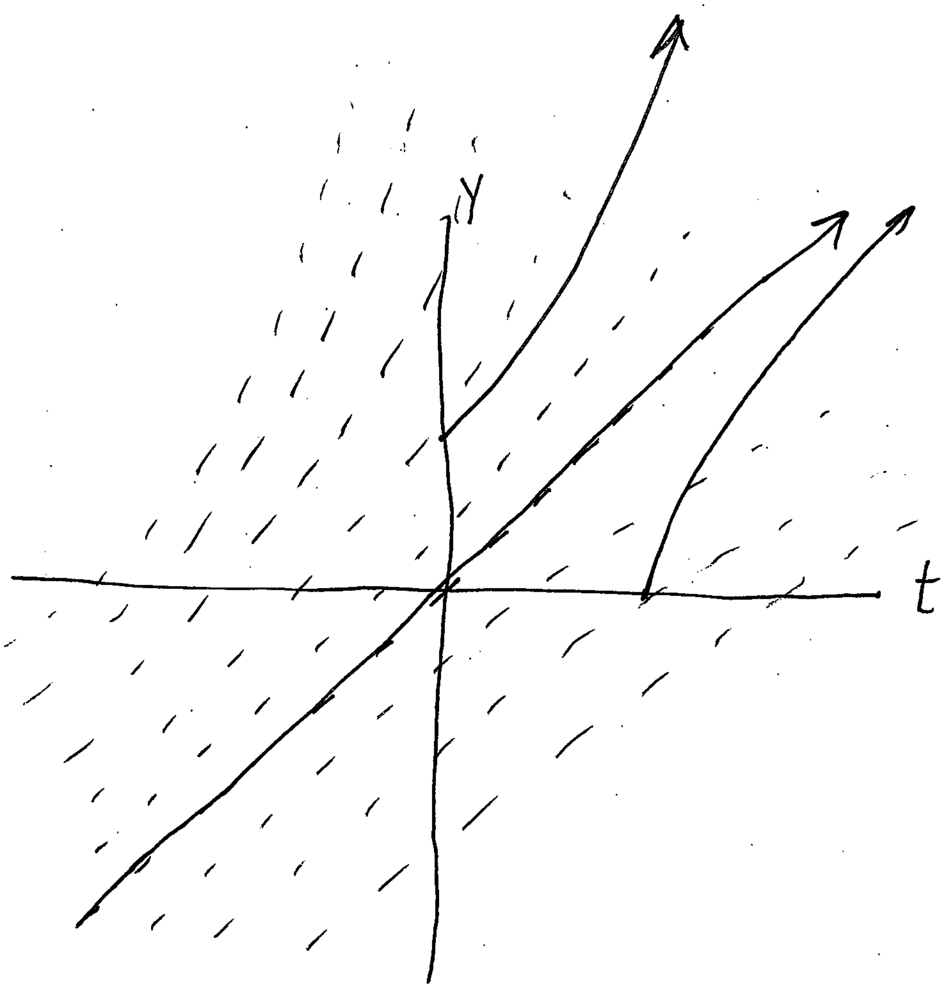
sketch representative solutions (i.e. $y(t)$). (*Hint: taking an indirect approach will likely prove most useful.*)

$$y' = (y-t)^2 + (y-t) + 1$$

Clearly if $y = t$, $\frac{dy}{dt} = 1$

Note; For $y > t$, $y - t > 0 \Rightarrow (y-t)^2 + (y-t) + 1 > 0 \Rightarrow y' > 0$

For $y < t$, $y - t < 0 \Rightarrow (y-t)^2 + y - t + 1 > 0$ because $(y-t)^2 > y-t$ for all $-1 < y - t < 0$



6. *(20 points)* Solve each of the following differential equations. *You may not use the same technique to solve both of them.*

(a) $\frac{dy}{dt} + 3y = t^2$

$Ly = t^2$

$Ly_h = 0$

$y_h = Ae^{-3t}$

$Ly_p = t^2$

$y_p = Ct^2 + Dt + E$

$y_p' = 2Ct + D$

$Ly_p = 3Ct^2 + 3Dt + 3E + 2Ct + D = t^2$

$3C = 1 \Rightarrow C = 1/3$

$3D + 2C = 0 \quad D = -2/9$

$3E + D = 0 \quad E = 2/27$

Check

$y_g = Ae^{-3t} + \frac{1}{3}t^2 - \frac{2t}{9} + \frac{2}{27}$

$Ly_g = -3Ae^{-3t} + \frac{2t}{3} - \frac{2}{9} + 3Ae^{-3t} + t^2 - \frac{2t}{3} + \frac{2}{9}$

$Ly_g \overset{\checkmark}{=} t^2$

(b) $\frac{dy}{dt} = ty - \pi t$

$\frac{dy}{dt} - ty = -\pi t$

$\mu = e^{\int -t\,dt}$

$= e^{-\frac{t^2}{2}}$

$e^{-\frac{t^2}{2}}\frac{dy}{dt} - tye^{-\frac{t^2}{2}} = -\pi te^{-\frac{t^2}{2}}$

$\frac{d}{dt}\left(e^{-\frac{t^2}{2}}y\right) =$

$e^{-\frac{t^2}{2}}y = \pi\int -te^{-\frac{t^2}{2}}dt$

$e^{-\frac{t^2}{2}}y = \pi e^{-\frac{t^2}{2}} + C$

$y = \pi + Ce^{+\frac{t^2}{2}}$

Check

$ty - \pi t \overset{?}{=} \left(\pi + Ce^{+\frac{t^2}{2}}\right)'$

$t(y - \pi) \overset{?}{=} tCe^{t^2/2}$

$t\left(Ce^{t^2/2}\right) \overset{?}{=} tCe^{t^2/2}$

$\checkmark$

7. *(15 points)* **QUICKIES!** Answers only. No partial credit.

a. Find one particular solution to the differential equation

$$\frac{dy}{dt} + 4y = 2.40.$$

$A + 4(At + B) = 2.4$

$\cancel{4A = 0.6}$

$A + 4B = 2.4$

$4A = 0$

$A = 0$

$B = 0.6$

$Ly_p = 2.4$

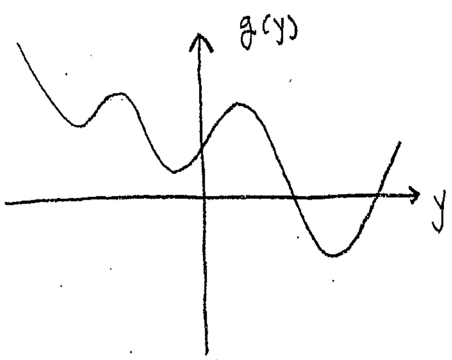
$\cancel{y_p = 2.40t}$ $\quad y_p = At + B$

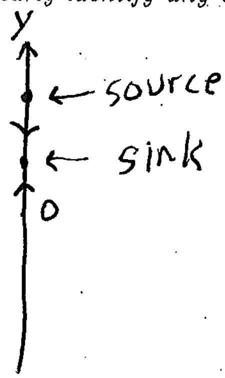
$\boxed{y_p = 0.6}$

b. Sketch the phase line that corresponds to the differential equation

$$\frac{dy}{dt} = g(y)$$

when the graph of $g(y)$ is as shown below. *On your phase line, clearly identify any equilibrium values.*

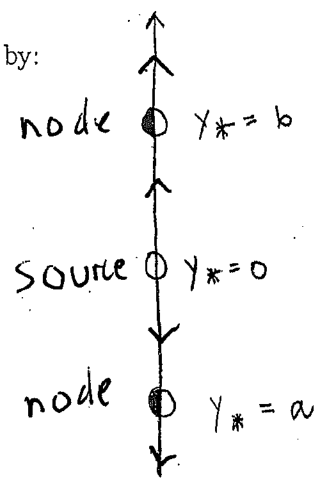


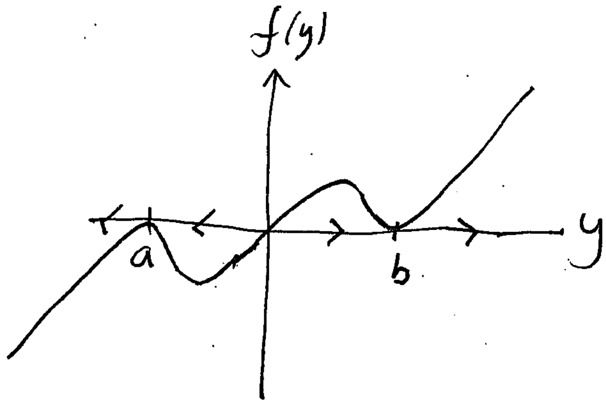


c. Sketch the graph of a possible function $f(y)$ for which the phase line of the differential equation

$$\frac{dy}{dt} = f(y)$$

is given by:





● sink

◐ node

○ source