FINAL EXAM: Differential Equations

Math 341 Fall 2008 Friday December 12

 1:00pm-4:00pm

Name: BUCKMIRE

Directions: Read *all* problems first before answering any of them. There are **17** pages in this test. This exam is split into two parts. **Part I** is a 1-hour exam on Chapter 6 (Laplace Transforms). **Part II** is a 1-hour cumulative exam on the central ideas, techniques and methods of the course. You have three hours to complete the entire exam. **No calculators.** You must show all relevant work to support your answers. Use complete English sentences as much as possible and CLEARLY indicate your final answers to be graded from your "scratch work."

You may consult a 8.5" by 11" "cheat sheet" with writing on both sides. There is a formula sheet at the end of Part 1 which includes all the formulas you should need for Laplace Transforms.

Pledge: I, ______________________________, pledge my honor as a human being and Occidental student, that I have followed all the rules above to the letter and in spirit.

No.	Score	Maximum
I.1		30
I.2		40
I.3		30
PART I (Chapter 6)		**100**
II.1		10
II.2		10
II.3		10
II.4		10
II.5		10
II.6		10
II.7		10
II.8		10
II.9		10
II.10		10
BONUS		10
PART II (Cumulative)		**100**
TOTAL		**200**

I.1. *[30 points total.]* **Convolution, Exponential Order, Duhamel's Principle.**
[10 points total.] **TRUE or FALSE.**
Are the following statements TRUE or FALSE – put your answer in the box. To receive ANY credit, you must also give a brief, and correct, explanation in support of your answer! For example, if you think the answer is **FALSE** providing a counterexample for which the statement is NOT TRUE is best. If you think the answer is **TRUE** you should prove why you think the statement is always true. Your explanation of your answer is worth FOUR TIMES as much as the answer you put in the box.

(a) **TRUE or FALSE?** "The convolution operation $\star$ possesses an identity element 1 such that $f(t) \star 1 = 1 \star f(t) = f(t)$ for every integrable function $f(t)$."

$\boxed{\text{FALSE}}$

$$1 \star f(t) = \int_0^t 1\, f(t-s)\,ds \neq f(t)$$

i.e. pick $f(t) = \sin(t)$ as a counterexample

(b) **TRUE or FALSE?** "Every integrable function $f(t)$ possesses a corresponding Laplace transform $F(s)$."

$\boxed{\text{FALSE}}$

If the function is not "of exponential order" it doesn't have a Laplace Transform, since the improper integral won't converge.

Pick $f(t) = \frac{1}{t^3}$ as a counter example

(c) **TRUE or FALSE?** "The exact solution $y(t)$ to the initial value problem $y'' + y = f(t)$, $y(0) = 0$, $y'(0) = 0$ can be obtained without computing $F(s)$, the Laplace transform of $f(t)$."

$\boxed{\text{TRUE}}$

$$(s^2+1)\,Y(s) = F(s)$$

$$Y(s) = F(s)\,\frac{1}{s^2+1}$$

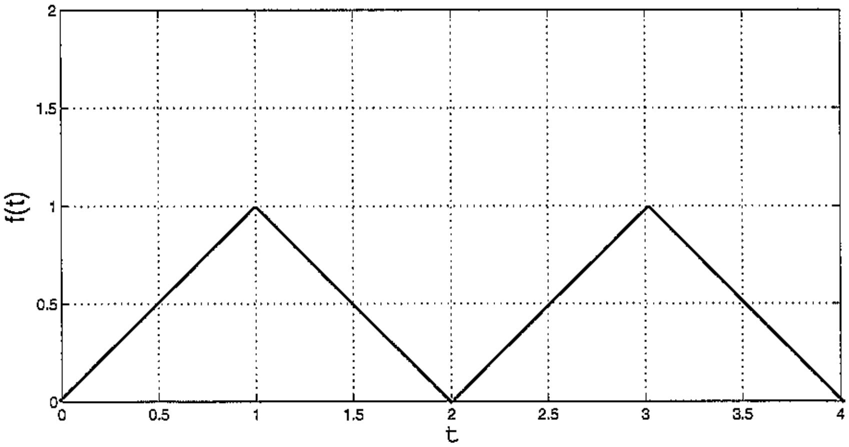
$$y(t) = \mathcal{L}^{-1}\left[F(s)\,\frac{1}{s^2+1}\right]$$

$$= f(t) \star \sin(t)$$

$$= \int^t f(s)\sin(t-s)\,ds$$

I.2. *[40 points total.]* **Laplace Transforms, Derivative/Anti-Derivatives of Laplace Transforms, Periodic Functions, Heaviside Function.**

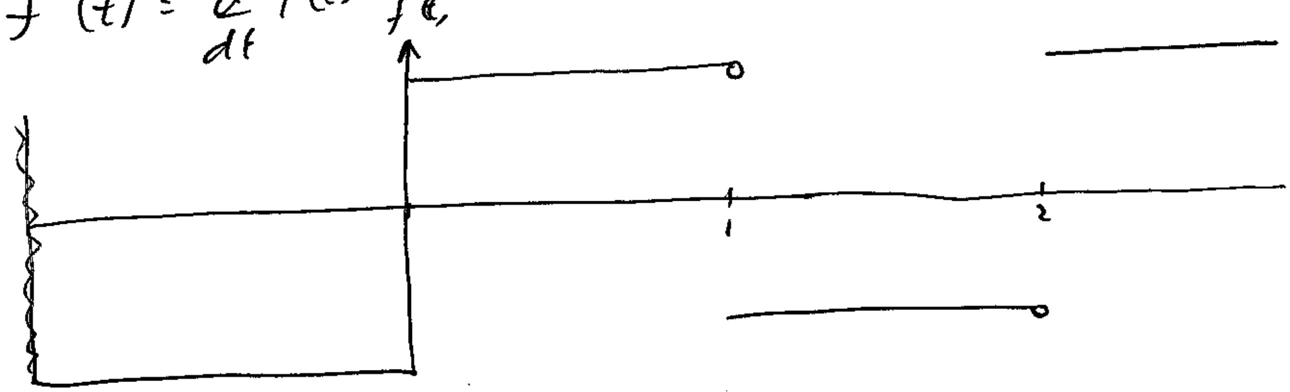
This problem is about showing that the Laplace Transform of the periodic triangular wave function $f(t)$ shown below is $F(s) = \frac{1}{s^2}\tanh\left(\frac{s}{2}\right)$.



(a) *[10 points]*. Show that $f'(t) = (-1)^{[t]} = \begin{cases} 1, & 0 \le t < 1, \\ -1, & 1 \le t < 2 \end{cases}$ where $f(t+2) = f(t)$ and $[t]$ is the greatest integer less than t and t is a positive real number. Provide a sketch of $f'(t)$ and give the period of $f'(t)$.

period is 2.

$f'(t) = \frac{d}{dt} f(t)$



(b) *[10 points]*. Given that $\sinh(s) = \frac{e^s - e^{-s}}{2}$, $\cosh(s) = \frac{e^s + e^{-s}}{2}$ and $\tanh(s) = \frac{\sinh(s)}{\cosh(s)}$ use algebra to show that $\frac{1 - e^{-s}}{s(1+e^{-s})} = \frac{1}{s}\tanh\left(\frac{s}{2}\right)$.

$$\tanh\left(\frac{s}{2}\right) = \frac{\sinh\left(\frac{s}{2}\right)}{\cosh\left(\frac{s}{2}\right)} = \frac{\frac{e^{s/2} - e^{-s/2}}{2}}{\frac{e^{s/2} + e^{-s/2}}{2}} = \frac{e^{s/2} - e^{-s/2}}{e^{s/2} + e^{-s/2}}$$

(divide by $e^{s/2}$ top & bottom)

$$= \frac{1 - e^{-s}}{1 + e^{-s}}$$

$$\frac{1}{s}\tanh\left(\frac{s}{2}\right) = \frac{1}{s}\left(\frac{1 - e^{-s}}{1 + e^{-s}}\right)$$

(c) *[10 points].* Using the fact that $f'(t)$ is periodic show that its Laplace Transform $\mathcal{L}[f'(t)] = \dfrac{1-e^{-s}}{s(1+e^{-s})}$.

$$\mathcal{L}[f'(t)] = \frac{1}{1-e^{-2s}} \int_0^2 f'(t) e^{-st}\,dt = \frac{1}{1-e^{-2s}}\left[\int_0^1 e^{-st}\,dt + \int_1^2 -1 e^{-st}\,dt\right]$$

$$= \frac{1}{1-(e^{-s})^2}\left[\left(\frac{e^{-st}}{-s}\right)\Big|_0^1 - \left(\frac{e^{-st}}{-s}\right)\Big|_1^2\right]$$

$$= \frac{1}{(1-e^{-s})(1+e^{-s})}\left[\left(\frac{e^{-s}}{-s} - \frac{1}{-s}\right) - \left(\frac{e^{-2s}}{-s} - \frac{e^{-s}}{-s}\right)\right]$$

$$= \frac{1}{(1-e^{-s})(1+e^{-s})}\left[\frac{1}{s} - \frac{2e^{-s}}{s} + \frac{e^{-2s}}{s}\right] = \frac{1}{(1-e^{-s})(1+e^{-s})}\frac{1}{s}(1-2e^{-s}+e^{-2s})$$

$$= \frac{1}{s}\frac{(1-e^{-s})^2}{(1-e^{-s})(1+e^{-s})} = \frac{1-e^{-s}}{1+e^{-s}}\cdot\frac{1}{s}$$

(d) *[10 points].* Using the results from (b) and (c), $\mathcal{L}[f'(t)] = \dfrac{1}{s}\tanh\left(\dfrac{s}{2}\right)$, find an expression for
$\mathcal{L}[f(t)] = F(s)$ when $f(t)$ is the periodic triangular wave function depicted on the previous page. EXPLAIN YOUR ANSWER.

$$\mathcal{L}[f'(t)] = \frac{1}{s}\tanh\left(\frac{s}{2}\right) \Rightarrow \mathcal{L}^{-1}\left[\frac{1}{s}\tanh\left(\frac{s}{2}\right)\right] = f'(t)$$

$$\mathcal{L}[f(t)] = \frac{1}{s^2}\tanh\left(\frac{s}{2}\right) \qquad \mathcal{L}^{-1}\left[\frac{\frac{1}{s}\tanh\left(\frac{s}{2}\right)}{s}\right] = f(t)$$

because $\mathcal{L}^{-1}\left[\dfrac{F(s)}{s}\right] = \displaystyle\int_0^t f'(\tau)\,d\tau = f(t) - f(0)$

in this case $f(0) = 0$

where $\mathcal{L}[f'(t)] = F(s)$

I.3. *[30 points total.]* **Convolution, Laplace Transforms of Anti-Derivatives, Inverse Laplace Transforms, Partial Fractions.**

Consider the expression $F(s) = \dfrac{1}{s^2(s-a)}$. There are several different methods to obtain the inverse Laplace Transform $f(t) = \mathcal{L}^{-1}[F(s)]$ for this function. You are asked to use two different techniques.

(a) *[15 points]*. Use a method to find $f(t)$ from $F(s)$. NAME YOUR METHOD.

CONVOLUTION THEOREM

$$\mathcal{L}^{-1}\left[\frac{1}{s^2}\cdot\frac{1}{s-a}\right] = \mathcal{L}^{-1}\left[\frac{1}{s^2}\right] \star \mathcal{L}^{-1}\left[\frac{1}{s-a}\right]$$

$$= t \star e^{at} = e^{at} \star t$$

$$= \int_0^t e^{a\tau}(t-\tau)\,d\tau = (t-\tau)\frac{e^{a\tau}}{a}\Big|_0^t - \int_0^t -\frac{e^{a\tau}}{a}\,d\tau$$

INTEG. Parts

$$u = t-\tau \qquad du = -1$$

$$dv = e^{a\tau} \qquad v = \frac{e^{a\tau}}{a}$$

$$= 0\cdot\frac{e^{at}}{a} - \frac{t}{a} + \frac{e^{a\tau}}{a^2}\Big|_0^t$$

$$= \frac{e^{at}}{a^2} - \frac{1}{a^2} - \frac{t}{a} = \boxed{\frac{1}{a^2}\left(e^{at} - at - 1\right) = \mathcal{L}^{-1}\left(\frac{1}{s^2}\frac{1}{s-a}\right)}$$

(b) *[15 points]*. Use a different method from the one you used in part (a) to find $f(t)$. NAME YOUR METHOD.

PARTIAL FRACTIONS

$$\frac{1}{s^2}\frac{1}{s-a} = \frac{As+B}{s^2} + \frac{C}{s-a} \Rightarrow 1 = (As+B)(s-a) + Cs^2$$

$$1 = As^2 + Bs - Aas - Ba + Cs^2$$

$$s = a \Rightarrow C = \frac{1}{a^2} \qquad A = -C = -\frac{1}{a^2}$$

$$s = 0 \qquad 1 = -aB \Rightarrow B = -\frac{1}{a}$$

$$s = -a$$

$$1 = \left(-Aa - \frac{1}{a}\right)(-2a) + \frac{1}{a^2}(a^2)$$

$$1 = 2a\left(Aa + \frac{1}{a}\right) + 1$$

$$Aa + \frac{1}{a} = 0 \Rightarrow A = -\frac{1}{a^2}$$

$$= -\frac{1}{a^2}\cdot\frac{1}{s} + -\frac{1}{a}\frac{1}{s^2} + \frac{1}{a^2}\frac{1}{s-a}$$

$$\mathcal{L}^{-1}\left[\frac{1}{s}\right] = 1 \quad \mathcal{L}^{-1}\left[\frac{1}{s^2}\right] = t \quad \mathcal{L}^{-1}\left[\frac{1}{s-a}\right] = e^{at}$$

$$\mathcal{L}^{-1}\left[\frac{1}{s^2}\frac{1}{s-a}\right] = -\frac{1}{a^2}\cdot 1 - \frac{1}{a}\cdot t + \frac{1}{a^2}e^{at} = \boxed{\left(e^{at} - at - 1\right)\frac{1}{a^2}}$$

3rd Method

DOUBLE INTEGRATION $F(s) = \frac{1}{s-a}$ $f(t) = e^{at}$

$$\frac{F(s)}{s^2} = \int_0^t\int_0^r e^{a\tau}\,d\tau\,dr \quad \int_0^t \frac{e^{a\tau}}{a}\Big|_0^r dr = \int_0^t \frac{e^{ar}}{a} - \frac{1}{a}\,dr = \frac{e^{ar}}{a^2} - \frac{r}{a}\Big|_0^t = \boxed{\frac{e^{at}}{a^2} - \frac{t}{a} - \frac{1}{a^2}}$$

II.1 *[10 points total.]* **Existence and Uniqueness Theorem.**

(a) Suppose $y' = f(x,y)$ where f and $\frac{\partial f}{\partial y}$ are continuous everywhere, i.e. at all (x,y) values. Does this mean that any solution $y(x)$ to the ODE must be continuous everywhere? EXPLAIN YOUR ANSWER.

f continuous everywhere $\Rightarrow$ $y(x)$ exists everywhere

f_y continuous everywhere $\Rightarrow$ $y(x)$ unique everywhere

Since y' exists, we know $y(x)$ is differentiable everywhere

differentiability implies continuity.

Yes, $y(x)$ <u>IS</u> continuous everywhere.

(b) Suppose $y' = xy^{1/3}$. What does the Existence and Uniqueness Theorem allow us to conclude about solutions to the ODE along $y = 0$? EXPLAIN YOUR ANSWER.

$f(x,y) = xy^{1/3}$ continuous along $y=0$

$f_y(x,y) = \frac{1}{3}xy^{-2/3}$ NOT continuous along $y=0$

Solutions along $y=0$ will exist but NOT be unique, i.e.

$\frac{dy}{dx} = xy^{1/3} \Rightarrow \frac{dy}{y^{1/3}} = x\,dx$

$y(A) = 0$

$y^{2/3} = \frac{x^2}{2} + C$

$x = A, y = 0 \Rightarrow 0 = \frac{A^2}{2} + C \Rightarrow C = -\frac{A^2}{2}$

$y^{2/3} = \frac{x^2 - A}{2}$

$y = \left[\frac{1}{2}(x^2 - A)\right]^{3/2}$ and $y = 0$

II.2 *[10 points total.]* **Bifurcation, Phase Lines, Equilibria, Stability.**
Consider the autonomous ordinary differential equation $y' = ay^2 + y^4$ where a is an arbitrary real-valued parameter. Find the equilibria values y^* and compute the bifurcation value a_B. Sketch phase lines for representative values $a < a_B$, $a = a_B$ and $a > a_B$. Clearly label any sinks, sources or nodes. Finally, sketch the bifurcation diagram in the ay^*-plane.

$$f(y) = (a+y^2)y^2 = 0 \Rightarrow y^* = 0 \text{ and } y^* = \pm\sqrt{-a}$$

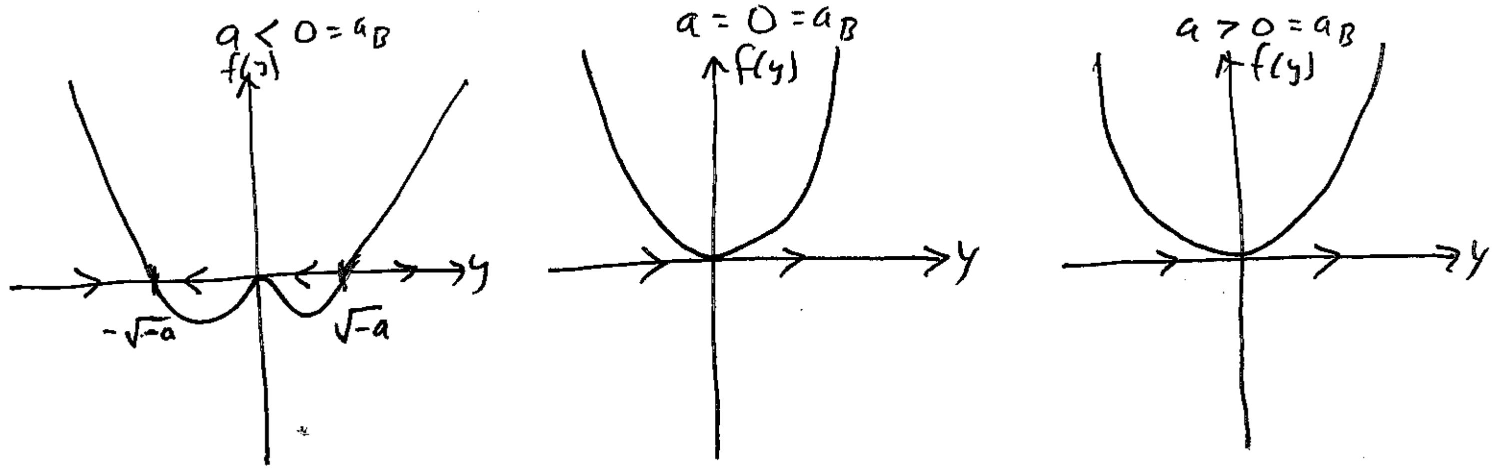
$$y^2 = 0 \text{ or } y^2 = -a$$

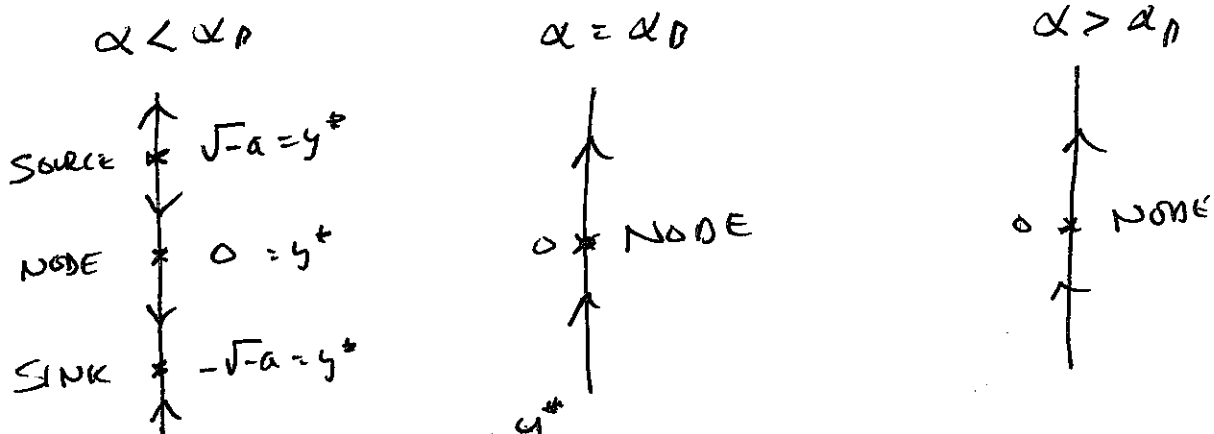
$$f_y = 2ay + 4y^3 = 0$$

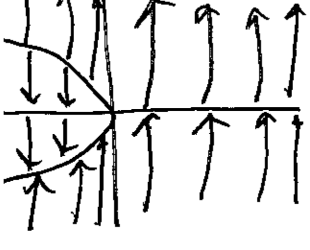
$$= 2y(a+2y^2) = 0 \Rightarrow y = 0 \text{ or } 2y^2 = -a$$

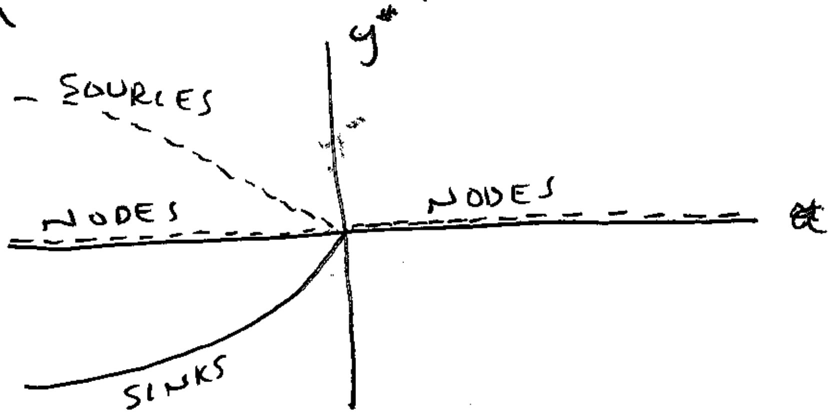
$$\text{but } y^2 = -a \Rightarrow 2a = a \Rightarrow a = 0 \checkmark$$

$$a_B = 0$$









II.3 *[10 points total.]* **Hamiltonian and Gradient Systems.**

Consider $\phi(x, y) = \frac{x^2}{2} - \frac{x^4}{4} - \frac{y^2}{2} + 8.$

(a) Write down a system of differential equations for which the given $\phi(x, y)$ is a Hamiltonian function. CONFIRM your given system is indeed a Hamiltonian system.

$$\dot{x} = f = \phi_y = -y$$

$$\dot{y} = g = -\phi_x = -x + x^3$$

$$f_x = 0 = -g_y \leftarrow$$ condition on Hamiltonian

(b) Write down a system of differential equations for which the given $\phi(x, y)$ is a Gradient function. CONFIRM your given system is indeed a Gradient system.

$$\dot{x} = f = \phi_x = x - x^3$$

$$\dot{y} = g = \phi_y = -y$$

$$f_y = 0 = g_x \leftarrow$$ condition on Gradient system

II.4 *[10 points total.]* **Equilibria of Linear Systems, Linearization, Jacobian, Eigenvalues.**

Consider the following nonlinear system of differential equations

$$\frac{dx}{dt} = -2x + 2x^2$$
$$\frac{dy}{dt} = -3x + y + 3x^2$$

Find and classify all the equlibrium points of the system.

$0 = -2x + 2x^2 \Rightarrow x = 0$ or 1

$0 = -3x + y + 3x^2 \Rightarrow$ when $x = 0, y = 0$

when $x = 1, y = 0$

$(0,0)$ and $(1,0)$ are the equilibrium points

$$J(x,y) = \begin{pmatrix} -2+4x & 0 \\ -3+6x & 1 \end{pmatrix}$$

$\lambda^2 + \lambda - 2 = 0$

$\cancel{\lambda^2 + 2\lambda + 2} = 0$

$$J(0,0) = \begin{pmatrix} -2 & 0 \\ -3 & 1 \end{pmatrix}$$

$\lambda = \frac{-1 \pm \sqrt{1^2 - 4(-2)(1)}}{2}$

$= \frac{-1 \pm \sqrt{9}}{2} = \frac{-1 \pm 3}{2}$

$= 1, -2$

eigenvalues are -2 and 1

so $(0,0)$ is a saddle

$$J(1,0) = \begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix}$$

Upper triangular matrix $\lambda = 2, 1$

So, $(1,0)$ is an unstable source.

II.5 *[10 points total.]* **Linear System of ODEs.**

Solve $\frac{d\vec{x}}{dt} = \begin{bmatrix} -2 & 3 \\ 3 & -2 \end{bmatrix} \vec{x}$.

$$\lambda^2 + 4\lambda - 5 = 0$$

$$(\lambda + 5)(\lambda - 1) = 0 \Rightarrow \lambda = 1, -5$$

$$E_1 = \cancel{\text{sp}}\ \text{null}(A - I) = \left(\begin{array}{cc|c} -3 & 3 & 0 \\ 3 & -3 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$x - y = 0$
y free

$$E_1 = \text{span}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$E_{-5} = \text{null}(A + 5I) = \left(\begin{array}{cc|c} 3 & 3 & 0 \\ 3 & 3 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$x + y = 0$
y free

$$E_{-5} = \text{Span}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\vec{x} = c_1 e^{t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-5t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

II.6 *[10 points total.]* **Nonhomogeneous and Homogeneous Problems.**

(a) Solve $t\dfrac{dy}{dt} + 2y = 0$. SHOW ALL YOUR WORK!

$$\frac{dy}{dt} = -\frac{2y}{t}$$

$$\frac{dy}{y} = -2\frac{dt}{t}$$

$$\ln y = -2\ln t + C$$

$$y = At^{-2} = \frac{A}{t^2}$$

$$y_h = \frac{A}{t^2}$$

(b) Solve $t\dfrac{dy}{dt} + 2y = 2t^2$, $\quad y(2) = 1$. SHOW ALL YOUR WORK!

$= y_h + y_p$

$$y(t) = -\frac{4}{t^2} + \frac{t^2}{2}$$

Guess $y_p = Bt^2$

$$ty_p' + 2y_p = 2t^2$$

$$t(2Bt) + 2(Bt^2) = 2t^2$$

$$2Bt^2 = t^2$$

$$\Rightarrow B = \frac{1}{2}$$

$$y = \frac{A}{t^2} + \frac{1}{2}t^2$$

$$y(2) = \frac{A}{4} + \frac{4}{2} = 1 \Rightarrow \frac{A}{4} = -1 \Rightarrow A = -4$$

or use Integrating factor!

$$y' + \frac{2}{t}y = 2t$$

$$\mu = e^{\int \frac{2}{t}dt} = e^{2\ln t} = t^2$$

$$t^2y' + 2ty = 2t^3$$

$$(t^2y)' = 2t^3$$

$$t^2y = 2\frac{t^4}{4} + C$$

$$y = \frac{t^2}{2} + \frac{C}{t^2}$$

$$1 = \frac{2^2}{2} + \frac{C}{2^2}$$

$$1 = 2 + \frac{C}{4}$$

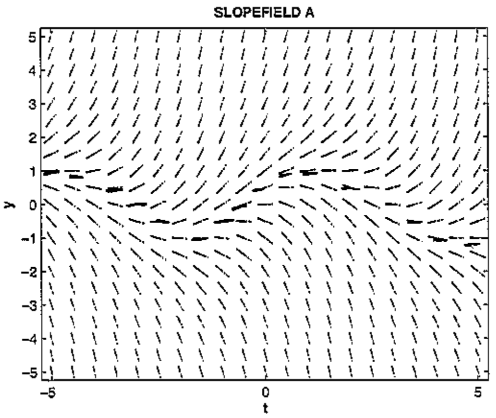
$$-1 = \frac{C}{4}$$

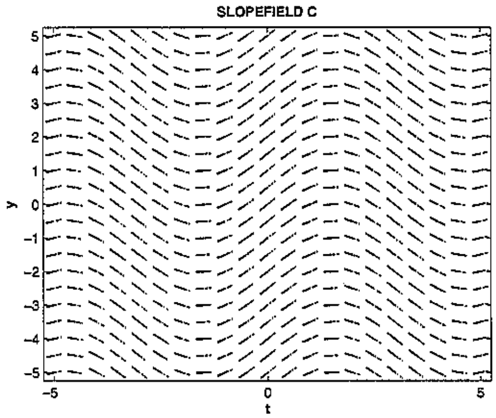
$$\boxed{y = \frac{t^2}{2} - \frac{4}{t^2}}$$

II.7 *[10 points total.]* **Phase Plane and Slope Fields.**
Consider the following first order differential equations:

1. $\frac{dy}{dt} = y - \sin(t)$ **2.** $\frac{dy}{dt} = \cos(t)$ **3.** $\frac{dy}{dt} = \cos(y)$ **4.** $\frac{dy}{dt} = t - \sin(y)$

Two slope fields are shown below. Match the slope fields with their associated ODE and provide an explanation. You will obtain four times as much credit for the explanation than for the correct slope field selection.





(**a**). The equation for slope field **A** is 1. $y' = y - \sin(t)$. This is because:

The slope field has slopes nearly equal to zero (or flat) along the curve $y = \sin(t)$.

A can't be 2 or 3 because it depends on t and y.

A can't be 4 because along $y = 0$ the slopes when $t < 0$ would have to all be negative. They are not.

(**b**). The equation for slope field **C** is 2. $\frac{dy}{dt} = \cos(t)$. This is because:

When t is constant (vertical line) the slopes are all the same, which is what we would expect for a slope field that depends only on t. 2 is the only choice.

3 would be the choice if the slopes remained constant along a horizontal ($y =$ constant) line

II.8 *[10 points total.]* **Heaviside Function and Delta Function.**

(a) Solve $y'' + y = \mathcal{H}(t), \quad y(0) = 0, \quad y'(0) = 0.$

$$(s^2+1)Y = \frac{1}{s}$$

$$Y = \frac{1}{s}\frac{1}{s^2+1} = \frac{1}{s} - \frac{s}{s^2+1}$$

$$y(t) = \mathcal{L}^{-1}\left(\frac{1}{s}\right) - \mathcal{L}^{-1}\left(\frac{s}{s^2+1}\right)$$

$$y(t) = 1 - \cos(t) = \int_0^t \sin\tau \, d\tau$$

$$= -\cos\tau \Big|_0^t$$

$$= -\cos t - (-1)$$

$$= 1 - \cos t$$

(b) Solve $y'' + y = \delta(t), \quad y(0) = 0, \quad y'(0) = 0.$

$$(s^2+1)Y = 1$$

$$Y = \frac{1}{s^2+1}$$

$$y(t) = \sin t$$

(c) Explain the difference between the two different physical situations represented by the two different initial value problems given in (a) and (b) and describe how the solutions to the two problems differ. One way to do this would be by giving a sketch of $y(t)$ for each proble,.

(a) is an input suddenly turning on at $t=0$,

(b) is a quick huge input to the system

II.9 *[10 points total.]* **Laplace Transforms and Initial Value Problems.**
Consider the zeroth-order Bessel's Equation, whose exact solution is the zeroth-order Bessel Function of the First Kind, denoted $J_0(t)$

$$ty'' + y' + ty = 0, \quad y(0) = 1, \quad y'(0) = 0.$$

Given that $\mathcal{L}[ty'] = -\frac{d}{ds}\{\mathcal{L}[y']\} = -\frac{d}{ds}\{sY(s) - y(0)\} = -s\frac{dY}{ds} - Y$, show that applying the Laplace Transform to the given initial value problem produces the following equation for $Y(s)$.

$$\mathcal{L}[ty'' + y' + ty] = -(s^2+1)\frac{dY}{ds} - sY = 0$$

$$\mathcal{L}[ty''] = -\frac{d}{ds}\mathcal{L}[y'']$$

$$= -\frac{d}{ds}\left[s^2Y - sy(0) - y'(0)\right]$$

$$= -\frac{d}{ds}\left[s^2Y - s\right]$$

$$= -s^2\frac{dY}{ds} - 2sY + 1$$

$$\mathcal{L}[y'] = sY - y(0) = sY - 1$$

$$\mathcal{L}[ty] = -\frac{d}{ds}Y$$

Add

$$-s^2Y' - 2sY + 1 + sY - 1 - Y' = 0$$

$$-(s^2+1)Y' - sY = 0 \quad \checkmark$$

$$(s^2+1)\frac{dY}{ds} + sY = 0$$

II.10 *[10 points total.]* **Separation of Variables.**
Consider the differential equation from **II.9**,

$$-(s^2+1)\frac{dY}{ds} - sY = 0$$

Let $Y(0) = A$ where A is a (un)known real value. Solve this initial value problem for $Y(s)$. (SEE BONUS PROBLEM to determine A!)

$$-(s^2+1)\frac{dY}{ds} = sY$$

$$\frac{dY}{Y} = \frac{-s}{s^2+1}\,ds$$

$$\int \frac{dY}{Y} = -\int \frac{s\,ds}{s^2+1}$$

$$\ln Y = -\frac{1}{2}\ln(s^2+1) + C$$

$$= -\ln\sqrt{s^2+1} + \ln B$$

$$\ln Y = \ln \frac{B}{\sqrt{s^2+1}}$$

$$Y = \frac{B}{\sqrt{s^2+1}}$$

$$Y(0) = B = A$$

$$Y(s) = \frac{A}{\sqrt{s^2+1}}$$

BONUS QUESTION *[10 points total.]*

(a) Given that $y(t)$ and its Laplace Transform $Y(s)$ satisfy the relationship

$$\lim_{s\to\infty} sY(s) = y(0)$$

use this result to obtain the unknown value of A from **II.10**. $y(0) = 1$

$$\lim_{s\to\infty} \frac{A}{\sqrt{s^2+1}} \cdot s = 1$$

$$\lim_{s\to\infty} \frac{A}{\frac{1 \cdot 2s}{2\sqrt{s^2+1}}} \overset{L'H}{=} 1$$

$$\lim_{s\to\infty} \frac{A\sqrt{s^2+1}}{s} =$$

$$\lim_{s\to\infty} A\sqrt{1+\frac{1}{s^2}} = 1$$

$$A \cdot 1 = 1$$

$$A = 1$$

(b) From **II.9**, **II.10** and part (a) above, what is an explicit functional form for the Laplace Transform of the Zeroth-Order Bessel's Function of the First Kind, i.e. $\mathcal{L}[J_0(t)]$?

$$\mathcal{L}[J_0(t)] = \frac{1}{\sqrt{s^2+1}}$$

$$F(s) = \mathcal{L}[f(t)] = \int_0^\infty f(t)e^{-st}dt$$

Laplace Transforms	
1	$\frac{1}{s}$
t^n	$\frac{n!}{s^{n+1}}$
$\frac{t^{m-1}}{(m-1)!}$	$\frac{1}{s^m}$
e^{at}	$\frac{1}{s-a}$
$\sin(kt)$	$\frac{k}{s^2+k^2}$
$\cos(kt)$	$\frac{s}{s^2+k^2}$
$\sinh(kt)$	$\frac{k}{s^2-k^2}$
$\cosh(kt)$	$\frac{s}{s^2-k^2}$
$\delta(t-a)$	e^{-as}
$\mathcal{H}(t-a)$	$\frac{e^{-as}}{s}$

Laplace Transform Formulas	
f'	$sF(s) - f(0)$
f''	$s^2F(s) - sf(0) - f'(0)$
$f^{(n)}$	$s^nF(s) - s^{(n-1)}f(0) - s^{(n-2)}f'(0) - \ldots - f^{(n-1)}(0)$
$e^{at}f(t)$	$F(s-a)$
$f(t-a)\mathcal{H}(t-a)$	$e^{-as}F(s)$
$t^n f(t)$	$(-1)^n \frac{d^n}{ds^n}F(s)$
$\int_0^t f(\tau)\ d\tau$	$\frac{F(s)}{s}$
$\int_0^t \int_0^r f(\tau)\ d\tau dr$	$\frac{F(s)}{s^2}$
$f * g$	$F(s)G(s)$
$\frac{f(t)}{t}$	$\int_s^\infty F(u)\ du$
$f(t+T) = f(t)$	$\frac{1}{1-e^{-Ts}} \int_0^T e^{-st}\ f(t)\ dt$