Math 341–Fall 2006

Final Exam–December 12th, 2006

Name: ____

You may use a calculator, your note card and something to write with. You must attach your notecard to the exam when you turn it in. You cannot use any differential equation solving or integrating capacities that are on your calculator. All work must be shown to receive full credit for any answer.

1. *(15 points)* Consider the following system of differential equations:

$$\begin{aligned}\frac{dx}{dt} &= -2x + 2x^2 \\ \frac{dy}{dt} &= -3x + y + 3x^2.\end{aligned}$$

(a) Find the nullclines of the system.

(b) Find the equilibrium points of the system.

(c) Classify the equilibrium points of the system.

(d) For each of the linearizations you performed in part *(c)*, find the eigenvectors associated with the linearized system.

(e) Briefly sketch the phase plane for this system–include the nullclines, label the equilibria, include the eigenvector solutions and draw reasonable trajectories based on your equilibrium point classification.

2. *(5 points)* Consider the two systems of differential equations:

$$\frac{dx}{dt} = 0.3x - 0.1xy \quad \frac{dx}{dt} = 0.3x - 3xy$$

$$\frac{dy}{dt} = -0.1y + 2xy \quad \frac{dy}{dt} = -2y + 0.1y.$$

Both of these are predator prey systems where the prey is given by $x(t)$ and the predator is given by $y(t)$. One of these systems involves very lethargic predators, those who seldom catch prey but who can live for a long time on a single prey (for example, boa constrictors). The other system refers to a very active predator that requires many prey to stay healthy (such as a small cat). The prey in each case is the same. Identify which system is which and justify your answer. (Your explanation only needs to be a few sentences.)

3. *(16 points)* Either find the general solution or solve the initial value problem (whichever is indicated).

1. $x\dfrac{dy}{dx} + 3y = x^3;\ \ y(1) = 2$

2. $\dfrac{d^2y}{dt^2} + 3y = 0$

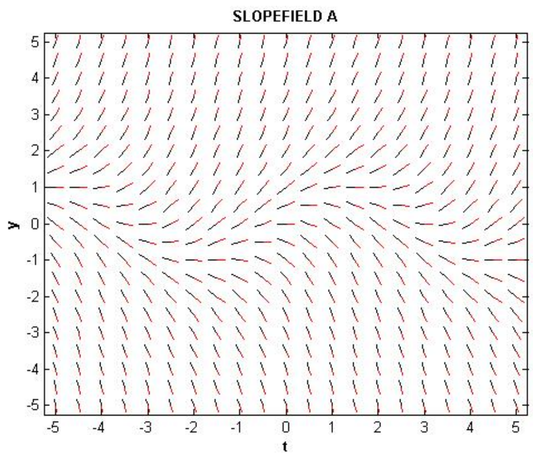
3. $y'' + 3y = sint.$

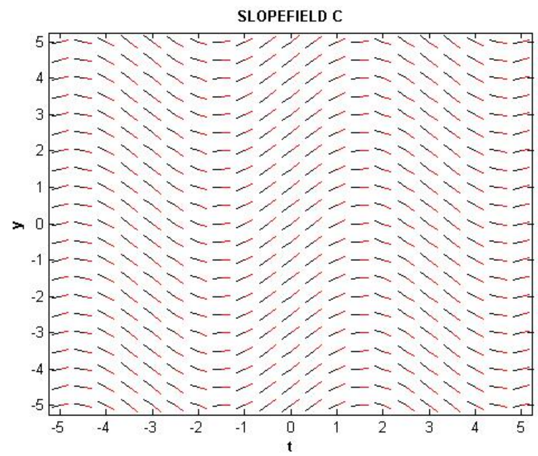
4. $\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 2 & 2 \\ -4 & 6 \end{pmatrix} \mathbf{Y}.$

4. *(10 points)* Consider the following first order equations:

$$1.\ \frac{dy}{dt} = y - \sin t \quad 2.\ \frac{dy}{dt} = \cos t \quad 3.\ \frac{dy}{dt} = \cos y \quad 4.\ \frac{dy}{dt} = t - \sin y$$

Two slope fields are shown below. Match the slope fields with their associated equation and provide an explanation. No explanation means *no credit*.





1. The equation for slope field **A** is __________. This is because:

2. The equation for slope field **C** is __________. This is because:

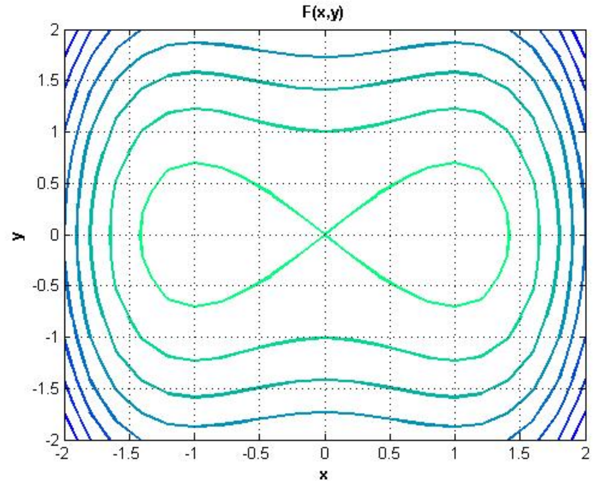
5. *(15 points) Sierpinski's Triangle!* Sierpinski's triangle is generated in the following way: Draw an equilateral triangle. This is S_0. Connect the midpoints of each side and then remove the "upside down" triangle; this is S_1. Repeat.

(a) Draw S_0 through S_3.

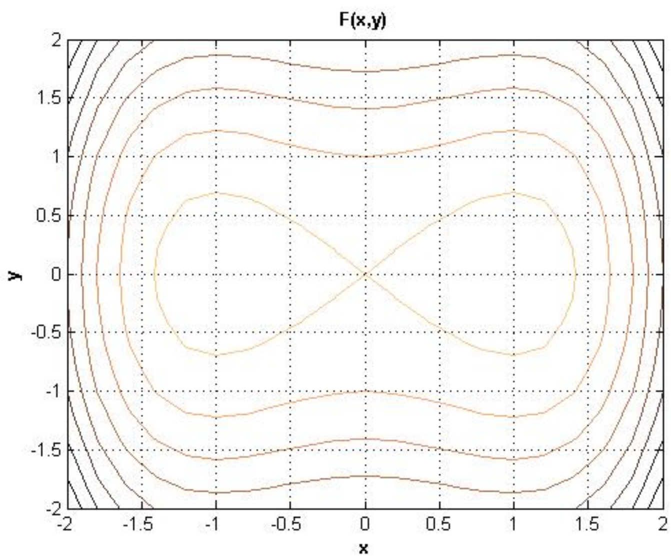
(b) Find the similarity dimension of S_∞, the Sierpinski triangle.

(c) Show that the Sierpinski triangle has no area.

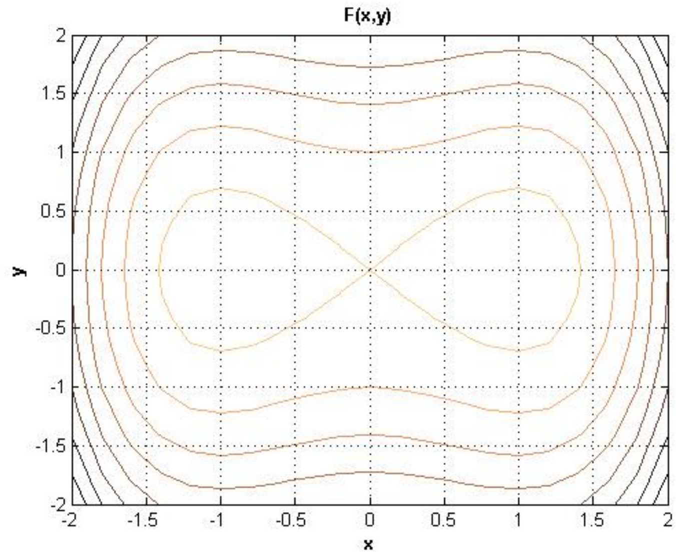
6. *(18 points)* The following are level curves for a function $F(x,y) = \frac{x^2}{2} - \frac{x^4}{4} - \frac{y^2}{2} + 8$. Refer to this figure to answer each part of this question.



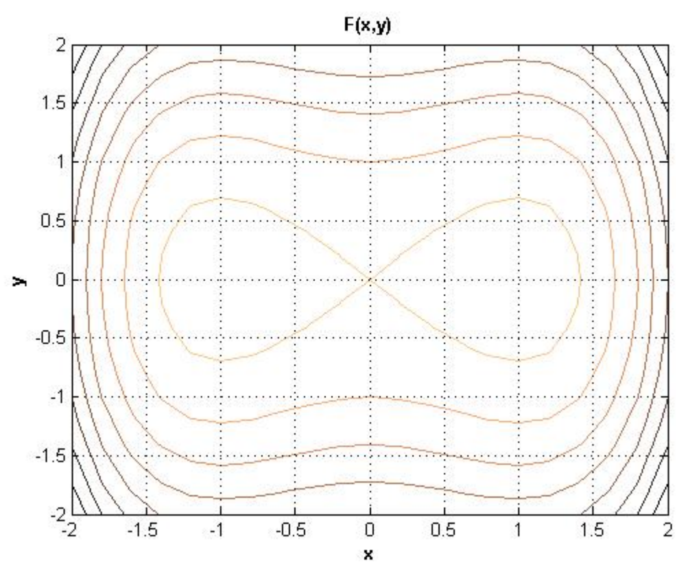
(a) Assume that $F(x,y)$ is a Hamiltonian function for a system of differential equations. Draw trajectories in the phase plane for the system, labeling and characterizing all equilibrium points. Then write down a system of differential equations for which this function could be a Hamiltonian.



(b) Assume that $F(x, y)$ is a Lyapunov function for a system of differential equations. Draw trajectories in the phase plane for the system, labeling and characterizing all equilibrium points.



(c) Assume that $F(x, y)$ is a gradient function for a system of differential equations. Draw trajectories in the phase plane for the system, labeling and characterizing all equilibrium points. Then write down a system of differential equations for which this function could be a gradient.



7. *(8 points)* Consider the following system of differential equations:

$$\begin{aligned} \frac{dx}{dt} &= y - y^3 \\ \frac{dy}{dt} &= -x - y^2 \end{aligned}$$

Predict the stability of the fixed point $(0, 0)$. $(0, 0)$ is indeed a center for this system. Show why.

8. *(8 points)* **Presentations!** The following are all multiple choice. You need to answer 6 questions **on topics which are not from your project!**. *(If you answer more than 6, you still must pick which 6 you want me to grade or else the first 6 will automatically be graded.)*

1. The *basic reproduction number* R_0 given by Rapatski and Klepac in their model of HIV/AIDS in Cuba is a threshold value for an epidemic. It comes from looking at the __________ of the system:
   (a) jacobian
   (b) eigenvalues
   (c) nullclines
   (d) isoclines.
2. How many different coordinate systems are used to describe the path of a bubble in the model by Shew and Pinton?
   (a) 1
   (b) 2
   (c) 3
   (d) 4
3. Modeling the evolution of an icicle is classified as a free-boundary problem because:
   (a) The boundary of the icicle is cost-effective.
   (b) The boundary of the icicle is not cone-shaped.
   (c) Heat transfer between the water and ice remains constant.
   (d) Heat transfer between the water and ice causes a shift in the boundary's location.
4. In the linguistic population model, the unilingual population is represented by x_1 and the bilingual population is represented by x_2. The assumption in the model that does not allow for an equilibrium of the form $x_1 = 0$ and $x_2 = k$, $k > 0$ is:
   (a) A unilingual population automatically produces a bilingual population.
   (b) Everyone should speak english.
   (c) A bilingual population automatically produces a unilingual population.
5. Which of the following questions were answered using Newton's Laws and o.d.e.s? (Circle *ALL* that apply.)
   (a) What is the escape velocity of a rocket?
   (b) What is the optimal landing velocity on the moon?
   (c) What type of rocket is most fuel efficient?
   (d) How does fuel consumption affect rocket speed?
6. Which of the following is *not* a parameter that arises in Lorenz' original system?
   (a) Rayleigh number
   (b) Prandtl number
   (c) Reynolds number

7. Which of the following is *not* a component in the STELLA modeling system?
   (a) stock
   (b) barrel
   (c) connector
   (d) flow
8. The graph of a solution to a model mass spring system with a rubber band attached differs from that of a typical mass spring system by which of the following? (Circle *ALL* that apply.)
   (a) amplitude
   (b) period
   (c) damping
   (d) lack of oscillations

9. *(12 points)* **Quickies!** No partial credit. Some fill-ins, some true false, some quick calculations.

(a) Write a differential equation which could match the following phase line:

(b) What kind of equilibria are not robust under linearization?

(c) **Fill-in:** Resonance occurs in a system when the natural frequency is the same as the ____________________.

(d) **True or False?** A linear system always has an equilibrium at the origin. (*No reasoning necessary.*)

(e) What are the two types of fixed points that occur on the repeated root parabola in the trace determinant plane?

(f) Give the general formula for Euler's Method with step size Δt to approximate solutions of $y' = F(y, t)$.

10. Essay! *(15 points)* In at least one page, explain what a bifurcation is and how it occurs in both one dimensional and two dimensional systems. Address representative bifurcations in one-dimension and explain how these compare to what goes on when dealing with a system of differential equations. Please be sure to be as grammatically correct as possible and take time to edit your response.