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# Differential Equations

Math 340 §2 Fall 2015

MWF 3:00-3:55pm Fowler 307

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## Worksheet 21

**TITLE** The Trace-Determinant Plane

**CURRENT READING** Blanchard, 3.7

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**Homework #10 Assignments due Monday November 2** (\* indicates EXTRA CREDIT)

**Section 5.1:** 3, 4, 5, 8, 18, 20\*.

**Section 5.3:** 2, 9, 12, 13, 14, 18\*.

**Chapter 5 Review:** 1, 2, 6, 7, 8, 9, 11, 12, 26, 27\*.

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### SUMMARY

We shall summarize all the possible equilibria one can get with a 2x2 linear system of ODEs into one big picture!

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### 1. Summarizing The Possibilities

Given a system of linear ODEs with associated matrix  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$  the characteristic polynomial is  $(a - \lambda)(d - \lambda) - bc = \lambda^2 - (a + d)\lambda + ad - bc = \lambda^2 - \text{tr}(A)\lambda + \det(A) = 0$ .

#### GROUPWORK

Your goal is to match the case # in the left column with the description of its critical point on the right (the list now is jumbled).

**CASE 1:** Real  $\lambda$ ,  $\lambda_1\lambda_2 < 0$

**A** Center

**CASE 2:** Real  $\lambda$ ,  $\lambda_1 \& \lambda_2 < 0$

**B** Spiral Source

**CASE 3:** Real  $\lambda$ ,  $\lambda_1 \& \lambda_2 > 0$

**C** (Stable) Node

**CASE 4:** Real  $\lambda$ ,  $\lambda_1 = \lambda_2 > 0$

**D** (Unstable) Node

**CASE 5:** Real  $\lambda$ ,  $\lambda_1 = \lambda_2 < 0$

**E** Saddle

**CASE 6:** Complex  $\lambda$ ,  $\text{Re}(\lambda) > 0$

**F** Spiral Sink

**CASE 7:** Complex  $\lambda$ ,  $\text{Re}(\lambda) < 0$

**G** Sink

**CASE 8:** Complex  $\lambda$ ,  $\text{Re}(\lambda) = 0$

**H** Source

Run the `LinearPhasePortraits` program from the Blanchard, Devaney & Hall textbook software. Use the slide bars to obtain different values of  $a$ ,  $b$ ,  $c$  and  $d$  and the different kinds of eigenvalues recorded above in the Cases. Record your results in the table below.

CASE #	a	b	c	d	$\lambda_1$	$\lambda_2$	Description
1							
2							
3							
4							
5							
6							
7							
8							

For more details, see the handout from **Edwards and Penney**, *Differential Equations*, **3rd Edition**, Prentice Hall: 2004, pp 381-389.

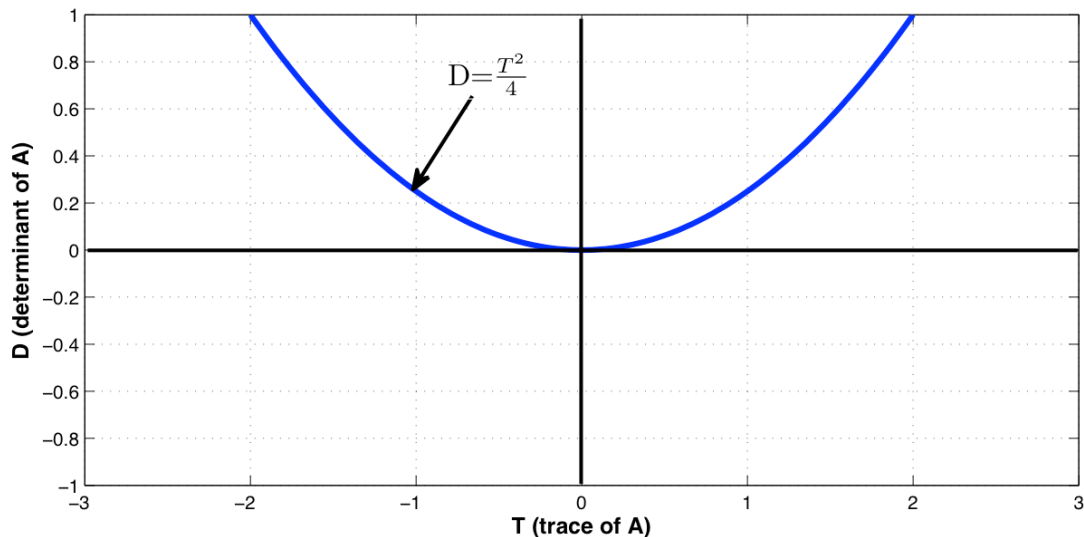
## 2. The Trace-Determinant Plane

Recall that the eigenvalues of a  $2 \times 2$  matrix are given by the roots of the polynomial  $p(\lambda) = \lambda^2 - \text{tr}(A)\lambda + \det(A) = 0$ .

It's also true that the trace of  $A$ , denoted  $\text{tr}(A)$  is equal to the **sum** of the eigenvalues  $\lambda_1 + \lambda_2$ . Let's use the symbol  $T$  for  $\text{tr}(A)$ . The determinant of  $A$ , denoted  $\det(A)$  is equal to the **product** of the eigenvalues  $\lambda_1\lambda_2$ . Let's use the symbol  $D$  for  $\det(A)$ .

Then we know that the eigenvalues are given by the solutions to  $\lambda^2 - T\lambda + D = 0$ , or 
$$\lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2}.$$

In other words, the condition on whether we will have real, complex or repeated eigenvalues depends on the behavior of the discriminant  $\Gamma = T^2 - 4D$ . See the figure drawn below. This is known as the **Trace-Determinant Plane**



This graph is an example of a parameter plane. As the matrix  $A$  changes it has different values of  $T$  and  $D$  and the linear system  $\frac{d\vec{x}}{dt} = A\vec{x}$  corresponding to that matrix will be located at a different location in  $(T, D)$ -space.

### Exercise

- (1) What kind of phase portraits will exist in  $(T, D)$ -space along the  $D$  axis?
- (2) What about the  $T$ -axis?
- (3) What kind of phase portraits occur along the curve  $D = \frac{T^2}{4}$ ?
- (4) What happens as one moves from the region just above the  $T$ -axis ( $D > 0$ ) to just below the  $T$ -axis ( $D < 0$ )? Does it matter if  $T > 0$  or  $T < 0$ ?
- (5) What kinds of solutions exist in the region above the parabola  $D = \frac{T^2}{4}$ ?