Differential Equations

Math 340 §2 Fall 2015 MWF 3:00-3:55pm Fowler 307

 http://sites.oxy.edu/ron/math/340/15/

Worksheet 20

TITLE Linear Systems with Repeated Eigenvalues
CURRENT READING Blanchard, 3.5

Homework #9 Assignments due Friday October 30
(* indicates EXTRA CREDIT)
Section 3.5: 3, 4, 9, 10, 12, 18*, 23*.
Section 3.7: 1,2*,6.
Chapter 3 Review: 3, 4, 6, 10, 13, 20*.

SUMMARY
We'll continue to explore the various scenarios that occur with linear systems of ODEs. This time dealing with those that possess repeated eigenvalues. This will involve the introduction of a new concepts, the Generalized Eigenvector. We will also review some important concepts from Linear Algebra, such as the Cayley-Hamilton Theorem.

1. Repeated Eigenvalues

Given a system of linear ODEs with associated matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ the characteristic polynomial is $p(\lambda) = (a-\lambda)(d-\lambda) - bc = \lambda^2 - \text{tr(A)}\lambda + \det(\text{A}) = 0$.

Previously we showed that the condition for repeated eigenvalues was $(a-d)^2 = -4bc$. In this case there will be only one solution to the quadratic equation, i.e. repeated eigenvalues equal to $\lambda = \dfrac{(a+d)}{2}$.

When there are two eiegenvalues and eigenvectors the general solution to $\dfrac{d\vec{x}}{dt} = A\vec{x}$ is $\vec{x} = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2$ where $A\vec{v}_1 = \lambda_1\vec{v}_1$ and $A\vec{v}_2 = \lambda_2\vec{v}_2$, i.e $\vec{v}_1$ and $\vec{v}_2$ are eigenvectors corresponding to eigenvalues λ_1 and λ_2.

The Easy Case
Q: What do we do if our one eigenvalue has two eigenvectors? (Is this even possible?)
A: As long as we have two eiegenvectors we can use the above formula for the general solution. In this case the problem is even simpler because if the eigenspace is 2-dimensional then every vector in $\mathbb{R}^2$ is an eigenvector so the easiest choice is $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. This situation is possible if the matrix has the form $\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{bmatrix}$.

The Hard Case
Q: So what do we do if we only have one eigenvalue λ (and only one eigenvector $\vec{v}$), i.e. $\vec{x}_1(t) = e^{\lambda t}\vec{v}$?
A: We need to find another vector function $\vec{x}_2(t)$ that is linearly independent to $\vec{x}_1(t)$ at every point t.
The answer turns out to be $\vec{x}_2(t) = e^{\lambda t}(\vec{w} + t\vec{v})$ where $(A - \lambda\mathcal{I})\vec{w} = \vec{v}$. In this formula $\vec{v}$ is an eigenvector of A and $\vec{w}$ is a **generalized eigenvector** of rank 2.

DEFINITION: **generalized eigenvector**

An eigenvector $\vec{w}$ associated with λ such that $(A-\lambda\mathcal{I})^r\vec{w}=\vec{0}$ but $(A-\lambda\mathcal{I})^{r-1}\vec{w}\neq\vec{0}$ is called a **generalized eigenvector of rank r**.

PROOF

Let's confirm that $\vec{x}(t)=e^{\lambda t}(\vec{w}+t\vec{v})$ is another solution to the ODE.

$$\begin{aligned}
\frac{d\vec{x}}{dt} &= A\vec{x}\\
\frac{d[e^{\lambda t}(\vec{w}+t\vec{v})]}{dt} &= A[e^{\lambda t}(\vec{w}+t\vec{v})]\\
\lambda e^{\lambda t}(\vec{w}+t\vec{v})]+e^{\lambda t}\vec{v} &= e^{\lambda t}[A\vec{w}+A\vec{v}t]\\
e^{\lambda t}(\lambda\vec{w}+\vec{v})+te^{\lambda t}\lambda\vec{v} &= e^{\lambda t}(A\vec{w})+(A\vec{v})te^{\lambda t}
\end{aligned}$$

Equating the $e^{\lambda t}$ terms produces the equation $\lambda\vec{w}+\vec{v}=A\vec{w}$, i.e. $\vec{v}=A\vec{w}-\lambda\vec{w}=(A-\lambda\mathcal{I})\vec{w}$
Equating the $te^{\lambda t}$ terms produces the equation $\lambda\vec{v}=A\vec{v}$

So, if we choose $\vec{v}$ and $\vec{w}$ to have these properties then $\vec{x}(t)=e^{\lambda t}(\vec{w}+t\vec{v})$ will solve $\frac{d\vec{x}}{dt}=A\vec{x}$. Yay! The general solution will be $\vec{x}=c_1e^{\lambda t}\vec{v}+c_2e^{\lambda t}(\vec{w}+t\vec{v})$.

RECALL

The Cayley-Hamilton Theorem states that a $n\times n$ matrix A satisfies its own characteristic polynomial. In other words, given $p(\lambda)=\det(A-\lambda\mathcal{I})=0$, $p(A)=\mathcal{O}$ where $\mathcal{I}$ is the $n\times n$ identity matrix and $\mathcal{O}$ is the $n\times n$ zero matrix. (What an awesome result!)

Since we know there is only one (repeated) eigenvalue λ, we know that the characteristic polynomial has the form $p(x)=(x-\lambda)^2=0$ which means that $p(A)=(A-\lambda\mathcal{I})^2=\mathcal{O}$.

$$\begin{aligned}
(A-\lambda\mathcal{I})^2 &= \mathcal{O} && \text{(From the Cayley-Hamilton Theorem)}\\
(A-\lambda\mathcal{I})^2\vec{w} &= \mathcal{O}\vec{w} && \text{(Multiply both sides by an unknown vector } \vec{w})\\
(A-\lambda\mathcal{I})[(A-\lambda\mathcal{I})\vec{w}] &= \vec{0} && \text{(Group terms and name the bracketed term } \vec{v})\\
(A-\lambda\mathcal{I})\vec{v} &= \vec{0} && \text{(Either } \vec{v}=\vec{0} \text{ or it's a generalized eigenvector of } A)
\end{aligned}$$

RECALL

The definition of an eigenvector is a vector $\vec{x}$ which lies in the nullspace of $A-\lambda\mathcal{I}$ (also known as the eigenspace E_λ), i.e. it solves the equation $(A-\lambda\mathcal{I})\vec{x}=\vec{0}$.

So from the Cayley-Hamilton Theorem we know that the vector $(A-\lambda\mathcal{I})\vec{w}$ lies in the one-dimensional eigenspace E_λ, i.e. it must be a scalar multiple of the non-zero eigenvector $\vec{v}$.
We still do not know the exact value of vector $\vec{w}$ but we can use the above information to compute it by solving the linear system $(A-\lambda\mathcal{I})\vec{w}=\vec{v}$.

Algebraic Multiplicity Is Less Than Or Equal To Geometric Multiplicty
Algebraic multiplicity of an eigenvalue is the number of times an eigenvalue satisfies the characteristic polynomial.
Geometric multiplicity of an eigenvector is the dimension of the corresponding eigenspace (or the number of eigenvectors corresponding to a particular eigenvalue).

Exercise

Given $A = \begin{bmatrix} -1 & -1 \\ 1 & -3 \end{bmatrix}$ find the eigenvalue(s) and eigenvector(s) of A and confirm that this matrix satisfies the Cayley-Hamilton Theorem.

EXAMPLE

We'll show that $\frac{d\vec{x}}{dt} = \begin{bmatrix} -1 & -1 \\ 1 & -3 \end{bmatrix} \vec{x}$ has the general solution $\vec{x}(t) = c_1 e^{-2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-2t} \left(\begin{bmatrix} 0 \\ -1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$.

Exercise

Consider the BONUS Question from **Exam #1, Fall 2009**. Find the general solution of $\frac{dx}{dt} = x + 3y, \quad \frac{dy}{dt} = y$. It turns out that the solution is $x(t) = c_1 e^t + c_2 t e^t$ and $y(t) = c_2 e^t$. We can confirm this result by writing our system in matrix form and using our formula for the general solution when repeated eigenvalues occur of $\vec{x} = c_1 e^{\lambda t} \vec{v} + c_2 e^{\lambda t} (\vec{w} + t\vec{v})$ where $\vec{v}$ is an eigenvector of the coefficient matrix and $\vec{w}$ is a generalized eigenvector.