Differential Equations

Math 340 §2 Fall 2015 MWF 3:00-3:55pm Fowler 307

 http://sites.oxy.edu/ron/math/340/15/

Worksheet 17

TITLE Straight Line Solutions
CURRENT READING Blanchard, 3.2

Homework #8 due Friday October 23 (* indicates EXTRA CREDIT)
Section 3.2: 8, 9, 12, 16*, 17, 18*.
Section 3.3: 3, 4, 7, 8, 20*.
Section 3.4: 1, 2, 3, 4, 16*, 23*.

SUMMARY
Eigenvalues and eigenvectors return from Linear Algebra and are important in the case where Linear Systems of ODEs have solutions that look like straight lines.

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1. The Significance of Eigenvectors and Eigenvalues

Recall the solutions $\vec{x}_1(t) = \begin{bmatrix} e^{2t} \\ 0 \end{bmatrix}$ and $\vec{x}_2(t) = \begin{bmatrix} -e^{-4t} \\ 2e^{-4t} \end{bmatrix}$ to the ODE $\frac{d\vec{x}}{dt} = \begin{bmatrix} 2 & 3 \\ 0 & -4 \end{bmatrix} \vec{x}$ from *Worksheet #16*.

Notice that $\vec{x}_1(t) = \begin{bmatrix} e^{2t} \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{2t}$ and $\vec{x}_2(t) = \begin{bmatrix} -e^{-4t} \\ 2e^{-4t} \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} e^{-4t}$.

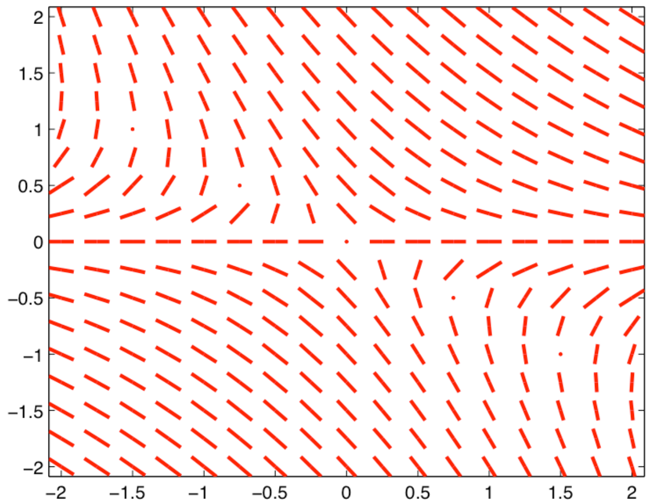
Question

Do you notice anything interesting about the vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$? Any relationship to the matrix $A = \begin{bmatrix} 2 & 3 \\ 0 & -4 \end{bmatrix}$? What happens if you multiply each vector by A?

Answer

The vectors in question are ______________________________.

Consider the direction field for the ODE $\frac{d\vec{x}}{dt} = \begin{bmatrix} 2 & 3 \\ 0 & -4 \end{bmatrix} \vec{x}$:



It turns out that the general solution to $\frac{d\vec{x}}{dt} = \begin{bmatrix} 2 & 3 \\ 0 & -4 \end{bmatrix} \vec{x}$ can be written as

$\vec{x} = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} -1 \\ 2 \end{bmatrix} e^{-4t}$.

Exercise

On the above direction field, we want to draw in the solutions $\vec{x}_1(t)$ and $\vec{x}_2(t)$. Does it matter what your initial condition is?

What happens as $t \to \infty$? What about as $t \to -\infty$ (i.e. reverse direction of the arrows)? Does one of the solutions seem more "attractive" than the other?

EXAMPLE

Consider the system $\frac{d\vec{x}}{dt} = \begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix} \vec{x}$. Find the eigenvalues λ and eigenvectors $\vec{v}$ of $\begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix}$.

Show that the general solution can be written as $\vec{x} = c_1 \vec{v}_1 e^{\lambda_1 t} + c_1 \vec{v}_2 e^{\lambda_2 t}$ and confirm that it is actually a solution of $\vec{x}' = \begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix} \vec{x}$.

2. General Solution To Homogeneous Linear Systems

THEOREM

The general solution $\vec{x}(t)$ on the interval $(-\infty, \infty)$ to a homogeneous system of linear DEs $\frac{d\vec{x}(t)}{dt} = A(t)\vec{x}(t)$ can be written as $\vec{x} = c_1 \vec{v}_1 e^{\lambda_1 t} + c_2 \vec{v}_2 e^{\lambda_2 t} + c_3 \vec{v}_3 e^{\lambda_3 t} + \ldots + c_n \vec{v}_n e^{\lambda_n t}$ where λ_1, λ_2, λ_3, ..., λ_n and $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$, ..., $\vec{v}_n$ are the eigenvalues and corresponding eigenvectors of the matrix A.

3. Phase Portraits With Straight Line Solutions

Exercise

Solve $\dfrac{dx}{dt} = 2x + 2y, \quad \dfrac{dy}{dt} = x + 3y.$

GroupWork

Use `HPGSystemSolver` (or `PPLANE`) to sketch the phase portrait of the linear system $\dfrac{d\vec{x}}{dt} = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} \vec{x}$ you solved above, in the space below.