Differential Equations

Math 340 §2 Fall 2015 | MWF 3:00-3:55pm Fowler 307

 | http://sites.oxy.edu/ron/math/340/15/

Worksheet 14

TITLE Existence and Uniqueness for Systems of 1st-Order ODEs
CURRENT READING Blanchard, Section 2.6

Homework Set #6 due Monday October 5
Section 2.1: 1, 2, 3, 5, 7, 10, 14*. **Section 2.2**: 7, 8, 11, 21* (EXPLAIN!), 24, 26.
Section 2.4: 2, 5, 7, 8. **Section 2.5**: 2, 3.
Chapter 2 Review: 2, 3, 7, 12, 13 15, 16, 20, 30*.

SUMMARY
Making a return in the context of n-dimensional systems of first-order differential equations $\dfrac{d\vec{x}}{dt} = \vec{F}(\vec{x})$ is the Existence and Uniqueness Theorem.

THEOREM

Consider a 2-dimensional system of first order differential equations of the form $\dfrac{d\vec{x}}{dt} = \vec{F}(\vec{x})$ where

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} \dfrac{dx}{dt} \\ \\ \dfrac{dx}{dt} \end{bmatrix} \text{ and } \vec{F}(\vec{x}) = \begin{bmatrix} f(x,y,t) \\ \\ g(x,y,t) \end{bmatrix}$$

Given that t_0 is an initial time and $\vec{x}_0$ is an initial value. If $\vec{F}(\vec{x})$ is **continuously differentiable** then there exists an $\epsilon > 0$ and a function $\vec{x}(t)$ defined for $t_0 - \epsilon \le t \le t_0 + \epsilon$ such that $\vec{x}(t)$ satisfies the initial value problem $\dfrac{d\vec{x}}{dt} = \vec{F}(\vec{x}), \quad \vec{x}(t_0) = \vec{x}_0$ and is unique for all t in this interval.

By continuously differentiable we mean that all the partial derivatives of f and g, i.e. f_x, f_y, f_t, g_x, g_y and g_t exists and are continuous on some open subset of the xy-plane.

Exercise

Suppose $\dfrac{d\vec{x}}{dt} = \vec{F}(\vec{x}) = \begin{bmatrix} y \\ -x + (1-x^2)y \end{bmatrix}$. Show that $\vec{F}(\vec{x})$ is everywhere continuously differentiable.

Implications of Existence and Uniqueness Theorem For Autonomous Systems
For autonomous systems of the form $\dot{x} = f(x, y), \quad \dot{y} = g(x, y)$ that we have generally been looking at the existence and uniqueness theorem means that no two solutions can go through the same point in the xy-plane at the same time.

However, we do know that for periodic solutions, the phase portrait will consists of a simple closed curve in the xy-plane.

Any other solution that starts in the interior of the closed curve represented by the periodic solution will be trapped in the interior of the solution curve represented by the periodic solution.

Any other solution that starts in the exterior of the closed curve represented by the periodic solution will be trapped outside the region in the xy-plane enclosed by the periodic solution.

EXAMPLE

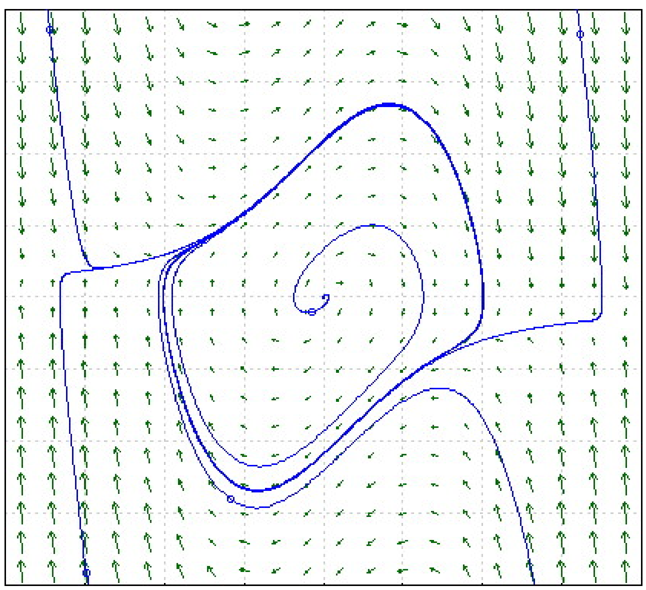
Consider the following famous equation, called the van der Pol oscillator model,

$$x'' + (x^2 - 1)x' + x = 0$$

which we can write as a system

$$\dot{x} = y, \qquad \dot{y} = -x + (1 - x^2)y$$

Looking at the Van der Pol system in `pplane` produces the following phase portrait.



GROUPWORK

What happens to all solutions regardless of where in the xy-plane they begin?