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# Differential Equations

Math 340 §2 Fall 2015  
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MWF 3:00-3:55pm Fowler 307  
<http://sites.oxy.edu/ron/math/340/15/>

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## *Worksheet 7*

**TITLE** Introduction to Bifurcations  
**CURRENT READING** Blanchard, 1.7

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### **Homework Set #4 due Friday September 18**

Section 1.6: 2, 7, 8, 19, 20, 30, 31, 41.

Section 1.7: 3, 6, 10, 13, 18.

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### **SUMMARY**

We will learn about a modern analytical technique which allows one to characterize how solutions of differential equations which contain parameters changes when the parameter values vary.

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#### **1. Parameter Sensitivity**

Consider a model for logistic growth of a fish population with constant harvesting given by the IVP  $P' = P(5 - P) - h$ ,  $P(0) = P_0$  where  $h \geq 0$ . Let's investigate how or if the solution changes as the values of the parameter  $h$  changes.

#### **GROUPWORK**

In the space below, draw phase lines for the critical points of the above IVP when the value of  $h$  equals 0, 2, 4, 6 and 8. Identify and classify any and all critical points for each value of  $h$ . What do you notice?

Is there a particular value of  $h$  for which the nature of the solution changes? If so, find it.

**DEFINITION: bifurcation**

A change in the qualitative nature of the phase portrait or long-term behavior of the solution of a differential equation when the value of a parameter changes is called a **bifurcation** of the DE. The value at which such changes occur is known as a **bifurcation point** or **bifurcation value** of the DE.

**DEFINITION: hyperbolic and nonhyperbolic critical points**

A critical point of an autonomous DE  $y' = f(y)$  is said to be **nonhyperbolic** if arbitrarily small changes (known as **perturbations**) in  $f(y)$  cause a bifurcation in the DE, i.e. critical points appear or disappear, or change the nature of their stability. If perturbations to  $f(y)$  cause changes in the quantitative but not qualitative nature of the critical points, these critical points are called **hyperbolic**.

**2. Analysis of Bifurcations****DEFINITION: bifurcation diagram**

A bifurcation diagram is a picture of the phase lines near a bifurcation value. It appears as a curve in the plane with the autonomous variable  $y$  on the vertical axis, and the bifurcation parameter on the horizontal axis. Generally a dotted line is used to indicate unstable sections of the curve (i.e. sources) and a solid line is used to indicate stable sections (i.e. sinks). A sink is denoted by a solid circle and a source by an empty circle. Nodes are depicted by half-filled in circles.

**EXAMPLE** Consider the one-parameter family of autonomous DE

$\frac{dy}{dt} = y^2 + \mu$ , where  $\mu$  is a parameter which can take on any real value. Let's sketch the **bifurcation diagram** of this DE.

This bifurcation is called a **saddle node bifurcation**. This is probably the most typical kind of bifurcation to arise.

**THEOREM**

Consider a one-parameter family of autonomous DEs where  $y' = f(y; \alpha)$  and  $\alpha$  is a parameter. The value  $\alpha_0$  will be a **bifurcation value** for the DE if and only if  $f(y_0; \alpha_0) = 0$  and  $f_y(y_0; \alpha_0) = 0$  simultaneously.

(NOTE: This is an ‘If and only If’ theorem which means the converse is true, i.e.  $A \Rightarrow B$  and  $B \Rightarrow A$  are both implied. Generally, definitions of quantities are always “If and only If” statements.)

**GROUPWORK**

Consider the following three different autonomous ODEs with an unknown real-valued parameter  $r$ . Draw bifurcation diagrams for each.

GROUP A:  $y' = ry - y^2$

GROUP B:  $y' = ry - y^3$

GROUP C:  $y' = ry + y^3$

These types of bifurcations are known as the **transcritical**, **supercritical pitchfork** and **subcritical pitchfork** bifurcations, respectively.

**Homework**

**Blanchard, page 106, #10.** For the one-parameter family  $y' = e^{-y^2} + \alpha$ , find the bifurcation values of  $\alpha$  and describe the bifurcation that takes place at each value. [HINT: Remember the Linearization Theorem!]