Differential Equations

(BY:) 2015 Ron Buckmire

Class 6: Wednesday September 9

TITLE Phase Lines and Equilibria
CURRENT READING Blanchard, 1.6

Homework Set #3 due Friday September 12
Section 1.4: 2, 6, 11, 15.
Section 1.5: 2, 3, 12, 14, 15.
Section 1.6: 2, 7, 8, 19, 20, 30, 31, 41.

SUMMARY
We will continue our qualitative analysis of differential equations by learning how to use **phase lines** and the classification of equilibrium points of autonomous, first-order ODEs.

DEFINITION: **critical point**

A **critical point** of an autonomous DE $y' = f(y)$ is a real number c such that $f(c) = 0$. Another name for critical point is **stationary point** or **equilibrium point**. If c is a critical point of an autonomous DE, then $y(x) = c$ is a constant solution of the DE.

DEFINITION: **phase portrait**

A **one dimensional phase portrait** of an autonomous DE $y' = f(y)$ is a diagram which indicates the values of the dependent variable for which y is increasing, decreasing or constant. Sometimes the vertical version of the phase portrait is called a **phase line**.

1. **Algorithm For Drawing A Phase Line**

   - Draw a vertical line
   - Find the equilibrium points (i.e. values such that $f(y) = 0$) and mark them on the line
   - Find intervals for which $f(y) > 0$ and mark them with up arrows $\uparrow$ or $\wedge$
   - Find intervals for which $f(y) < 0$ and mark them with down arrows $\downarrow$ or $\vee$

The textbook likes to have you think of the phase line as a rope with people moving up and down the rope in the directions the arrows are pointing to visualize solutions dynamically.

EXAMPLE

Consider the autonomous differential equation $\dfrac{dy}{dt} = y(a - by)$ where $a > 0$ and $b > 0$.
1 Find the critical points of the DE.

2 Determine the values of y for which $y(t)$ is increasing and decreasing

3 Draw the phase line for this DE

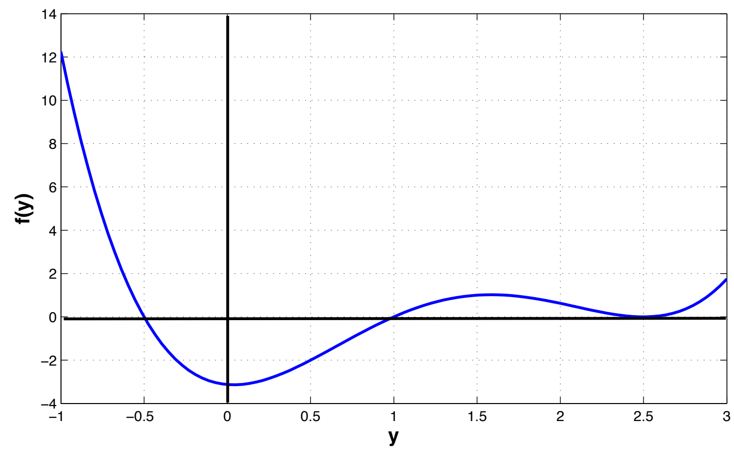
2. Obtaining Solution Information from Phase Lines

Consider $y' = f(y)$ where $f(y)$ is a continuously differentiable function and $y(t)$ is a solution to an autonomous ordinary differential equation. The following conclusions can be made

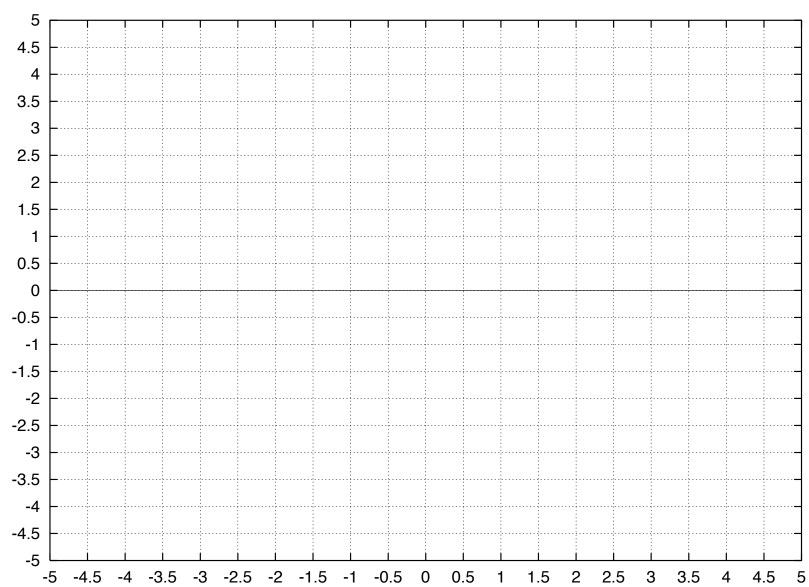
- If $f(y(0)) = 0$ then $y(t) = y(0)$ for all t and $y(0)$ is an equilibrium point
- If $f(y(0)) > 0$ then $y(t)$ is increasing for all t and either $y(t) \to \infty$ as t increases or $y(t)$ tends to the first equilibrium point larger than $y(0)$
- If $f(y(0)) < 0$ then $y(t)$ is decreasing for all t and either $y(t) \to -\infty$ as t increases or $y(t)$ tends to the first equilibrium point smaller than $y(0)$

Exercise

Draw the phase line in the space on the left for the corresponding ODE $y' = f(y)$ where $f(y)$ versus y is graphed below to the right.



Draw graphs of various particular solutions starting at $y(0) = -1$, $y(0) = 0$, $y(0) = 1$, $y(0) = 2$ and $y(0) = 3$ in the ty-plane given below.



3. Classifying Equilibrium Points: Sink, Source or Node

A critical value c is a point where $y' = f(c) = 0$ splits an interval into two different regions. So there are four possible scenarios for the behavior near c: $(+, 0, +)$, $(+, 0, -)$, $(-, 0, +)$ and $(-, 0, -)$.

EXAMPLE

Draw the phase line for each of these cases and then classify the corresponding critical points as asymptotically stable (i.e. attractor or **sink**), unstable (repellor or **source**) or semi-stable **node**.

Exercise

Draw graphs of the autonomous function $f(y)$ near equilibrium points classified as a sink, source or node corresponding to the phase line in the example above

THEOREM: Linearization Theorem

IF y_0 is an equilibrium point of the autonomous differential equation $y' = f(y)$ where $f(y)$ is a continuously differentiable function, THEN

- If $f'(y_0) < 0$ then y_0 is a sink.
- If $f'(y_0) > 0$ then y_0 is a source.
- If $f'(y_0) = 0$ then more information is needed to classify the equilibrium point.

EXAMPLE

What can you say about the equilibrium point of the ODE $y' = y(\cos(y^5 + 2y) - 27\pi y^4)$ at $y = 0$?

Exercise

Inspired by **Blanchard, Devaney & Hall, #43, page 93**.
Suppose $y' = f(y)$ and $y = y_0$ is an equilibrium point and

(a) $f'(y_0) = 0$, $f''(y_0) > 0$: Is y_0 a sink, source or node? EXPLAIN.

(b) $f'(y_0) = 0$, $f''(y_0) < 0$: Is y_0 a sink, source or node? EXPLAIN.

(c) $f'(y_0) = 0$, $f''(y_0) = 0$ and $f'''(y_0) > 0$: Is y_0 a sink, source or node? EXPLAIN.

(d) $f'(y_0) = 0$, $f''(y_0) = 0$ and $f'''(y_0) < 0$: Is y_0 a sink, source or node? EXPLAIN.