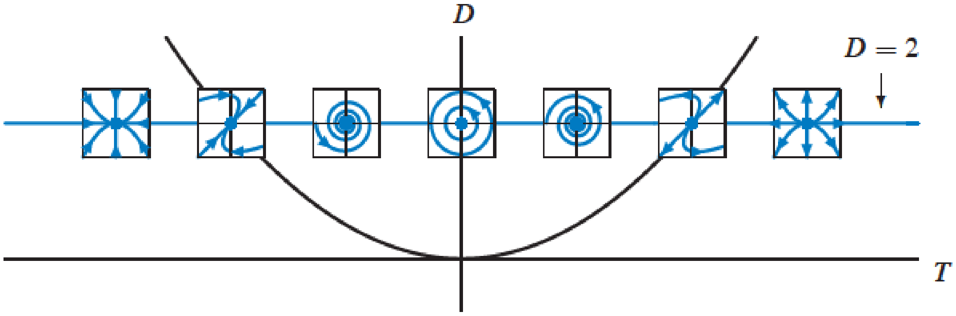
1. Table 3.2

Possibilities for linear systems

type	condition on λ	examples
sink	$\lambda_1 < \lambda_2 < 0$	Sec. 3.7, Fig. 3.52
saddle	$\lambda_1 < 0 < \lambda_2$	Sec. 3.3, Fig. 3.12–3.14
source	$0 < \lambda_1 < \lambda_2$	Sec. 3.3, Fig. 3.19
spiral sink	$\lambda = \alpha \pm i\beta, \alpha < 0, \beta \neq 0$	Sec. 3.1, Fig. 3.2 and 3.4
spiral source	$\lambda = \alpha \pm i\beta, \alpha > 0, \beta \neq 0$	Sec. 3.4, Fig. 3.29–3.30
center	$\lambda_1 = \pm i\beta, \beta \neq 0$	Sec. 3.1, Fig. 3.1 and 3.3
		Sec. 3.4, Fig. 3.28
sink	$\lambda_1 = \lambda_2 < 0$	Sec. 3.5, Fig. 3.35–3.36
(special case)	One line of eigenvectors	
source	$0 < \lambda_1 = \lambda_2$	Sec. 3.5, Ex. 2
(special case)	One line of eigenvectors	
sink	$\lambda_1 = \lambda_2 < 0$	Sec. 3.5, Ex. 23
(special case)	Every vector is eigenvector	
source	$0 < \lambda_1 = \lambda_2$	Sec. 3.5, Ex. 23
(special case)	Every vector is eigenvector	
no name	$\lambda_1 < \lambda_2 = 0$	Sec. 3.5, Fig. 3.39–3.40
no name	$0 = \lambda_1 < \lambda_2$	Sec. 3.5, Ex. 19
no name	$\lambda_1 = \lambda_2 = 0$	Sec. 3.5, Ex. 21
	One line of eigenvectors	
no name	$\lambda_1 = \lambda_2 = 0$	entire plane of equilibrium points
	Every vector is an eigenvector	

2. **(a)**



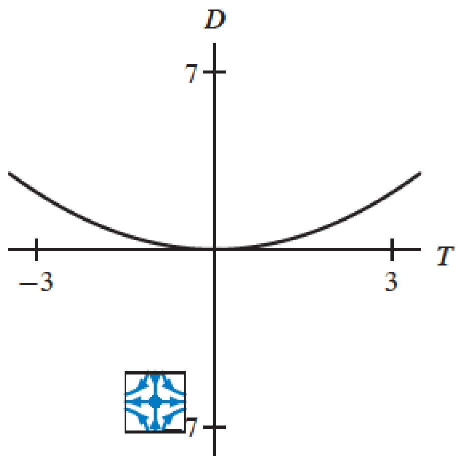
(b) The curve in the trace-determinant plane is the horizontal line given by $D = 2$. The eigenvalues are the roots of $\lambda^2 - a\lambda + 2$, which are

$$\frac{a}{2} \pm \frac{\sqrt{a^2 - 8}}{2}.$$

So we have complex eigenvalues if $|a| < 2\sqrt{2}$, real eigenvalues if $|a| > 2\sqrt{2}$, and repeated eigenvalues if $a = \pm 2\sqrt{2}$. Glancing at the trace-determinant plane, we see that we have a sink with real eigenvalues if $a < -2\sqrt{2}$, a spiral sink if $-2\sqrt{2} < a < 0$, and spiral source if $0 < a < 2\sqrt{2}$, and a source with real eigenvalues if $a > 2\sqrt{2}$.

(c) Bifurcations occur at $a = -2\sqrt{2}$, where we have a sink with repeated eigenvalues, at $a = 2\sqrt{2}$, where we have a source with repeated eigenvalues, and at $a = 0$, where we have a center.

6. **(a)**

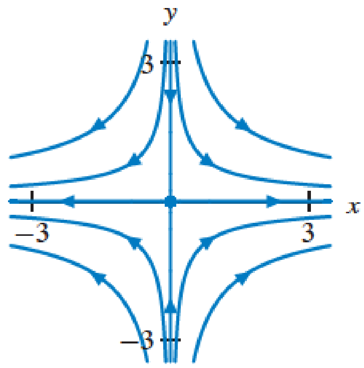


(b) The curve in the trace-determinant plane is not a curve at all. For all values of a, $T = -1$ and $D = -6$. So the curve is simply a point in the trace-determinant plane. For all a, we have a saddle.

(c) There are no bifurcations, since the origin is always a saddle. (There is nothing special about $a = 0$, by the way.)

3. The system has eigenvalues -2 and 3. One eigenvector associated with $\lambda = 3$ is $(1, 0)$, and one eigenvector associated with $\lambda = -2$ is $(0, 1)$. The general solution is

$$\mathbf{Y}(t) = k_1 e^{3t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + k_2 e^{-2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$



4. By definition, the zero vector, $\mathbf{Y}_1$, is never an eigenvector. We can check the others by computing $\mathbf{AY}$. For example,

$$\mathbf{AY}_2 = \mathbf{A} \begin{pmatrix} 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix} = \mathbf{Y}_2,$$

so $\mathbf{Y}_2$ is an eigenvector (with eigenvalue $\lambda = 1$). On the other hand,

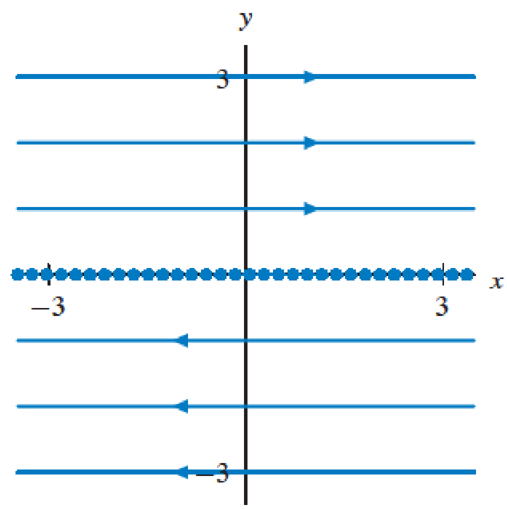
$$\mathbf{AY}_3 = \begin{pmatrix} 1 \\ 5 \end{pmatrix},$$

which is not a scalar multiple of $\mathbf{Y}_3$, so $\mathbf{Y}_3$ is not an eigenvector. Also, $\mathbf{AY}_4 = 3\mathbf{Y}_4$, so $\mathbf{Y}_4$ is an eigenvector (with eigenvalue $\lambda = 3$). Since we know that $\mathbf{Y}_2$ is an eigenvector and $\mathbf{Y}_5 = -2\mathbf{Y}_2$, $\mathbf{Y}_5$ is also an eigenvector. The vectors $\mathbf{Y}_2$ and $\mathbf{Y}_4$ are two linearly independent eigenvectors corresponding to different eigenvalues. Therefore, $\mathbf{Y}_6$ cannot be an eigenvector because it is neither a scalar multiple of $\mathbf{Y}_2$ nor $\mathbf{Y}_4$.

6. Written in coordinates, the system is $dx/dt = 0$ and $dy/dt = x - y$. Hence, the equilibrium points are all points on the line $y = x$.

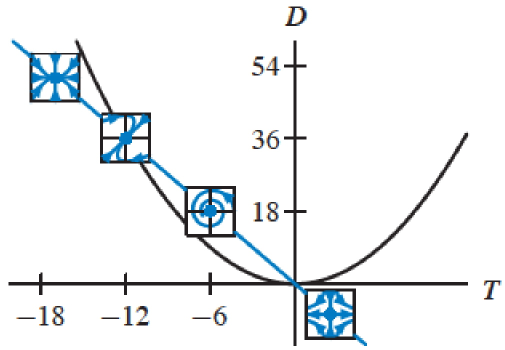
10. Written in terms of coordinates, the system is $dx/dt = y$ and $dy/dt = 0$. From the second equation, we see that $y(t) = k_2$, where k_2 is an arbitrary constant. Then $x(t) = k_2 t + k_1$, where k_1 is another arbitrary constant. In vector notation, the general solution is

$$\mathbf{Y}(t) = \begin{pmatrix} k_2 t + k_1 \\ k_2 \end{pmatrix}.$$



13. True. Linear systems have solutions that consist of just sine and cosine functions only when the eigenvalues are purely imaginary (that is, of the form $\pm i\omega$). In this case, the sine and cosine terms are of the form $\sin \omega t$ and $\cos \omega t$. For the first coordinate of $\mathbf{Y}(t)$ to be part of a solution, we would have to have $\omega = 2$, but the second coordinate would force $\omega = 1$. So this function cannot be the solution of a linear system.

20. **(a)** The trace T is a, and the determinant D is $-3a$. Therefore, the curve in the trace-determinant plane is $D = -3T$.



(b) The line $D = -3T$ crosses the parabola $T^2 - 4D = 0$ at two points—at $(T, D) = (-12, 36)$ if $a = -12$ and at $(T, D) = (0, 0)$ if $a = 0$. Therefore, bifucations occur at $a = -12$ and at $a = 0$. The portion of the line for which $a < -12$ corresponds to a positive determinant and a negative trace such that $T^2 - 4D < 0$. The corresponding phase portraits are real sinks. If $a = -12$, we have a sink with repeated eigenvalues. If $-12 < a < 0$, we have complex eigenvalues with negative real parts. Therefore, the phase portraits are spiral sinks. If $a = 0$, we have a degenerate case where the y-axis is an entire line of equilibrium points. Finally, if $a > 0$, the corresponding portion of the line is below the T-axis, and the phase portraits are saddles.

3. **(a)** The linearized system is

$$\frac{dx}{dt} = -2x + y$$
$$\frac{dy}{dt} = -y.$$

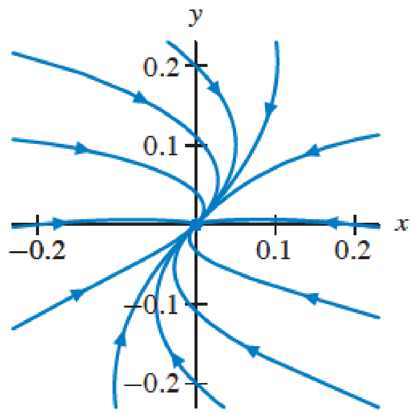
We can see this either by "dropping higher-order terms" or by computing the Jacobian matrix

$$\begin{pmatrix} -2 & 1 \\ 2x & -1 \end{pmatrix}$$

and evaluating it at $(0, 0)$.

(b) The eigenvalues of the linearized system are -2 and -1, so $(0, 0)$ is a sink.

(c) The vector $(1, 0)$ is an eigenvector for eigenvalue -2 and $(1, 1)$ is an eigenvector for the eigenvalue -1.



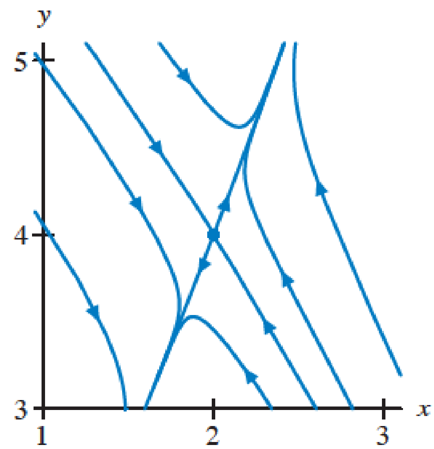
(d) By computing the Jacobian matrix

$$\begin{pmatrix} -2 & 1 \\ 2x & -1 \end{pmatrix}$$

and evaluating at $(2, 4)$, we see that linearized system at $(2, 4)$ is

$$\frac{dx}{dt} = -2x + y$$
$$\frac{dy}{dt} = 4x - y.$$

Its eigenvalues are $(-3 \pm \sqrt{17})/2$, so $(2, 4)$ is a saddle.



4. **(a)** The equilibrium points occur where the vector field is zero, that is, at solutions of

$$\begin{cases} -x = 0 \\ -4x^3 + y = 0. \end{cases}$$

So, $x = y = 0$ is the only equilibrium point.

(b) The Jacobian matrix of this system is

$$\begin{pmatrix} -1 & 0 \\ -12x^2 & 1 \end{pmatrix},$$

which at $(0, 0)$ is equal to

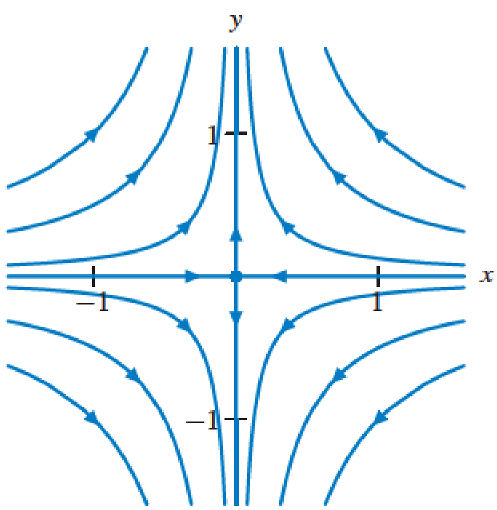
$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

So the linearized system at $(0, 0)$ is

$$\frac{dx}{dt} = -x$$
$$\frac{dy}{dt} = y$$

(we could also see this by "dropping the higher order terms").

(c) The eigenvalues of the linearized system at the origin are -1 and 1, so the origin is a saddle. The linearized system decouples, so solutions approach the origin along the x-axis and tend away form the origin along the y-axis.



5. **(a)** Using separation of variables (or simple guessing), we have $x(t) = x_0 e^{-t}$.

(b) Using the result in part (a), we can rewrite the equation for dy/dt as

$$\frac{dy}{dt} = y - 4x_0^3 e^{-3t}.$$

This first-order equation is a nonhomogeneous linear equation.

The general solution of its associated homogeneous equation is ke^t. To find a particular solution to the nonhomogeneous equation, we rewrite it as

$$\frac{dy}{dt} - y = -4x_0^3 e^{-3t},$$

and we guess a solution of the form $y_p = \alpha e^{-3t}$. Substituting this guess into the left-hand side of the equation yields

$$\frac{dy_p}{dt} - y_p = -4\alpha e^{-3t}.$$

Therefore, y_p is a solution if $\alpha = x_0^3$. The general solution of the original equation is

$$y(t) = x_0^3 e^{-3t} + ke^t.$$

To express this result in terms of the initial condition $y(0) = y_0$, we evaluate at $t = 0$ and note that $k = y_0 - x_0^3$. We conclude that

$$y(t) = x_0^3 e^{-3t} + (y_0 - x_0^3)e^t.$$

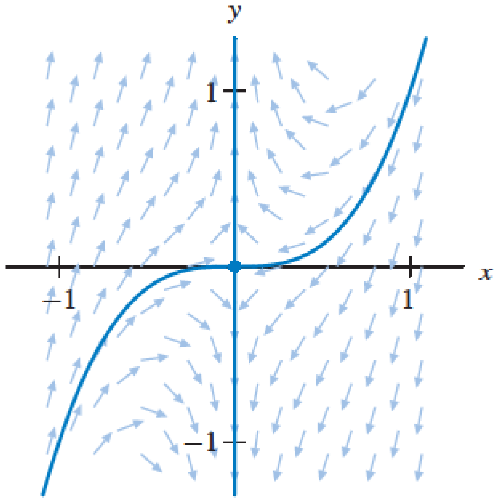
(c) The general solution of the system is

$$\begin{aligned} x(t) &= x_0 e^{-t} \\ y(t) &= x_0^3 e^{-3t} + (y_0 - x_0^3)e^t. \end{aligned}$$

(d) For all solutions, $x(t) \to 0$ as $t \to \infty$. For a solution to tend to the origin as $t \to \infty$, we must have $y(t) \to 0$, and this can happen only if $y_0 - x_0^3 = 0$.

(e) Since $x = x_0e^{-t}$, we see that a solution will tend toward the origin as $t \to -\infty$ only if $x_0 = 0$. In that case, $y(t) = y_0e^t$, and $y(t) \to 0$ as $t \to -\infty$.

(f)



(g) Solutions tend away from the origin along the y-axis in both systems. In the nonlinear system, solutions approach the origin along the curve $y = x^3$ which is tangent to the x-axis. For the linearized system, solutions tend to the origin along the x-axis. Near the origin, the phase portraits are almost the same.

8. **(a)** The equilibrium points are $(0, 0)$, $(0, 30)$, and $(10, 0)$. To determine the type of each equilibrium point, we compute the Jacobian matrix, which is

$$\begin{pmatrix} -2x - y + 10 & -x \\ -2y & -2x - 2y + 30 \end{pmatrix},$$

and evaluate it at the point. At $(0, 0)$, the Jacobian is

$$\begin{pmatrix} 10 & 0 \\ 0 & 30 \end{pmatrix},$$

and the eigenvalues are 10 and 30. Thus, the origin is a source. At $(0, 30)$, the Jacobian matrix is

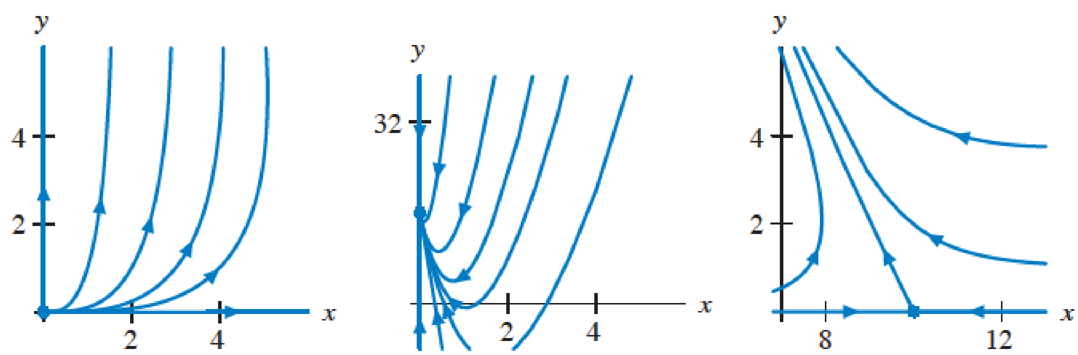
$$\begin{pmatrix} -20 & 0 \\ -60 & -30 \end{pmatrix},$$

and the eigenvalues are -20 and -30. So $(0, 30)$ is a sink. The Jacobian at $(10, 0)$ is

$$\begin{pmatrix} -10 & -10 \\ 0 & 10 \end{pmatrix},$$

and the eigenvalues are -10 and 10. Therefore, $(10, 0)$ is a saddle.

(b)



18. **(a)** The equation $x^2 - a = 0$ has no solutions if $a < 0$.

(b) The equilibrium points are $(\pm\sqrt{a}, 0)$.

(c) When $a = 0$, the only equilibrium point is $(0, 0)$.

(d) The Jacobian matrix is

$$\begin{pmatrix} 2x & 0 \\ -2xy & -x^2 - 1 \end{pmatrix}.$$

At $(0, 0)$, the Jacobian matrix is

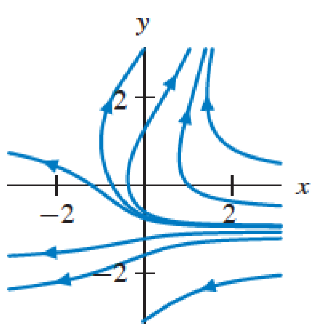
$$\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix},$$

which has eigenvalues -1 and 0. So $(0, 0)$ is a node.

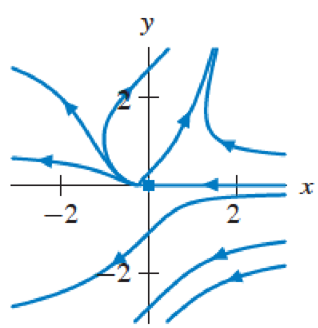
20. **(a)** The equilibrium points are $(\pm\sqrt{a}, a)$, so there are no equilibrium points if $a < 0$, one equilibrium point if $a = 0$, and two equilibrium points if $a > 0$

(b) If $a = 0$, the equilibrium point at the origin has eigenvalues 0 and 1 and is a node. If $a > 0$, the system has two equilibrium points, a saddle at $(\sqrt{a}, a)$ with eigenvalues $-2\sqrt{a}$ and 1 and a source at $(-\sqrt{a}, a)$ with eigenvalues $2\sqrt{a}$ and 1. A bifurcation occurs at $a = 0$ because the number of equilibrium points changes. It also reasonable to say that there is a bifurcation at $a = 1/4$ because the source at $(-\sqrt{a}, a)$ has repeated eigenvalues. For all other positive values of a, these eigenvalues are real and distinct.

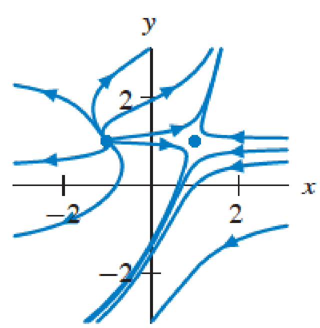
(c) Note that for all values of the parameter a, the line $y = a$ is invariant. If $a < 0$, all solutions come from and go to infinity. If $a = 0$, most solutions come from and go to infinity, but there are separatrices associated to the equilibrium point at the origin. If $a > 0$, some solutions come from and go to infinity, but many solutions come from the source at $(-\sqrt{a}, a)$ and go to infinity. There is also a separating solution along the line $y = a$ that comes from the source at $(-\sqrt{a}, a)$ and goes to the saddle at $(\sqrt{a}, a)$.



Phase portrait for $a < 0$



Phase portrait for $a = 0$



Phase portrait for $a > 0$