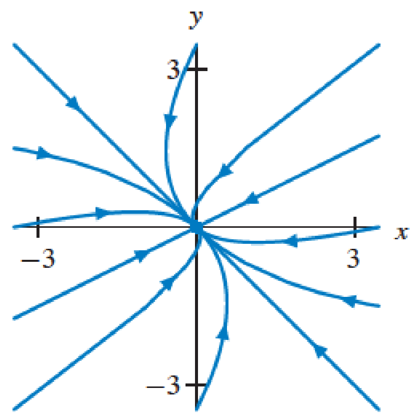
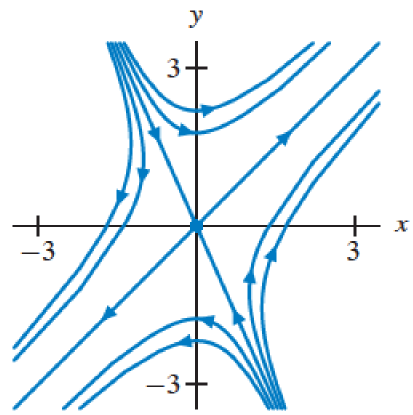
3. As we computed in Exercise 3 of Section 3.2, the eigenvalues are $\lambda_1 = -3$ and $\lambda_2 = -6$. The eigenvectors (x_1, y_1) for the eigenvalue $\lambda_1 = -3$ satisfy $y_1 = -x_1$, and the eigenvectors for $\lambda_2 = -6$ satisfy $x_2 = 2y_2$. The equilibrium point at the origin is a sink.



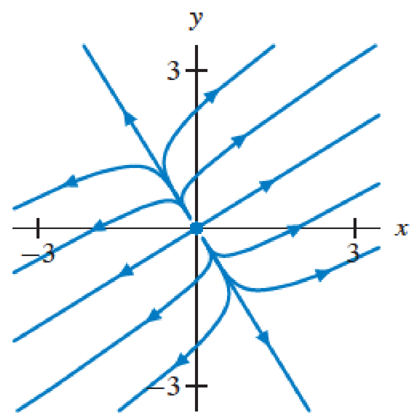
4. As we computed in Exercise 6 of Section 3.2, the eigenvalues are $\lambda_1 = -4$ and $\lambda_2 = 9$. The eigenvectors (x_1, y_1) for the eigenvalue $\lambda_1 = -4$ satisfy $9x_1 = -4y_1$, and the eigenvectors (x_2, y_2) for $\lambda_2 = 9$ satisfy the equation $y_2 = x_2$. The equilibrium point at the origin is a saddle.



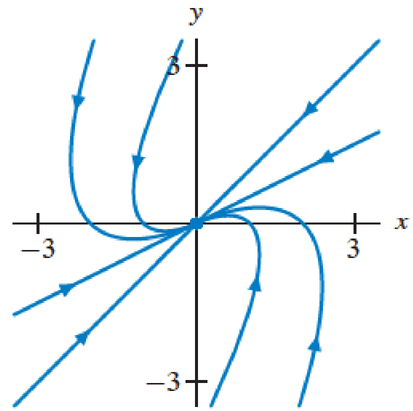
7. As we computed in Exercise 9 of Section 3.2, the eigenvalues are

$$\lambda_1 = \frac{3+\sqrt{5}}{2} \quad \text{and} \quad \lambda_2 = \frac{3-\sqrt{5}}{2}.$$

The eigenvectors (x_1, y_1) for the eigenvalue λ_1 satisfy $y_1 = (-1+\sqrt{5})x_1/2$, and the eigenvectors (x_2, y_2) for λ_2 satisfy $y_2 = (-1-\sqrt{5})x_2/2$. The equilibrium point at the origin is a source.



8. As we computed in Exercise 10 of Section 3.2, the eigenvalues are $\lambda_1 = -2$ and $\lambda_2 = -3$. The eigenvectors (x_1, y_1) for the eigenvalue $\lambda_1 = -2$ satisfy $x_1 = 2y_1$, and the eigenvectors (x_2, y_2) for $\lambda_2 = -3$ satisfy $x_2 = y_2$. The equilibrium point at the origin is a sink.



20. **(a)** The characteristic equation is

$$(2-\lambda)(-2-\lambda)-12=\lambda^2-16=0,$$

so the eigenvalues are $\lambda_1=-4$ and $\lambda_2=4$. Therefore, the equilibrium point at the origin is a saddle.

(b) To find all the straight-line solutions, we must calculate the eigenvectors. For the eigenvalue $\lambda_1=-4$, we have the simultaneous equations

$$\begin{cases} 2x_1+6y_1=-4x_1 \\ 2x_1-2y_1=-4y_1, \end{cases}$$

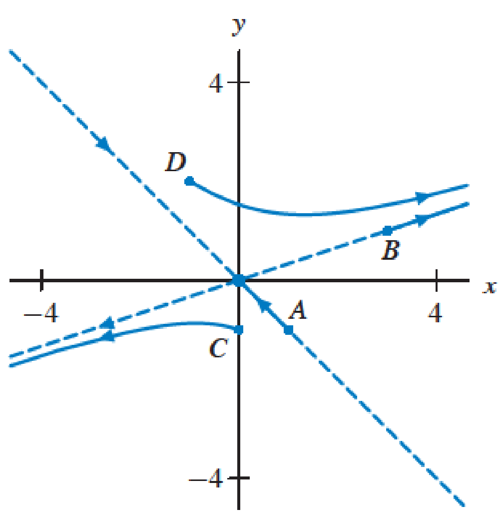
and we obtain $y_1=-x_1$. In other words, all vectors on the line $y_1=-x_1$ are eigenvectors for λ_1. Therefore, any solution of the form $e^{-4t}(x_1,-x_1)$ for any x_1 is a straight-line solution corresponding to the eigenvalue $\lambda_1=-4$.

To calculate the eigenvectors associated to the eigenvalue $\lambda_2=4$, we must solve the equations

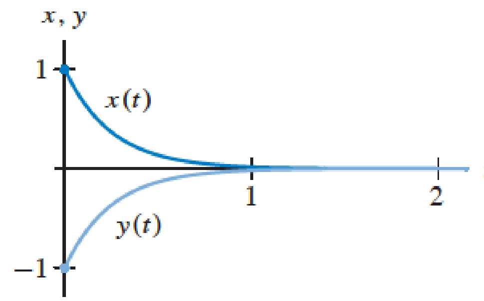
$$\begin{cases} 2x_2+6y_2=4x_2 \\ 2x_2-2y_2=4y_2, \end{cases}$$

and we obtain $x_2=3y_2$. Therefore, any solution of the form $e^{4t}(3y_2,y_2)$ for any y_2 is a straight-line solution corresponding to the eigenvalue $\lambda_2=4$.

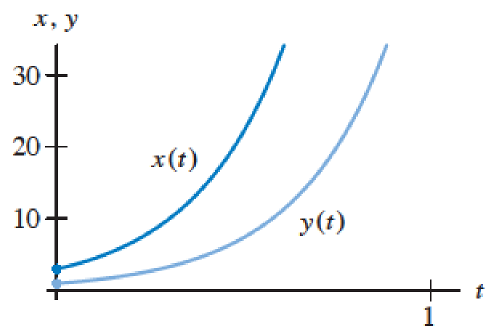
(c) In the phase plane, the only solution curves that approach the origin are those whose initial conditions lie on the line $y=-x$. All other solution curves eventually approach those that correspond to the line $x=3y$.



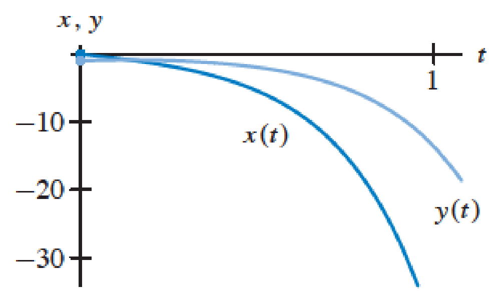
The initial condition $A = (1, -1)$ lies on the line $y = -x$. Therefore, it corresponds to a straight-line solution. In fact, the formula for its solution is $e^{-4t}(1, -1)$.



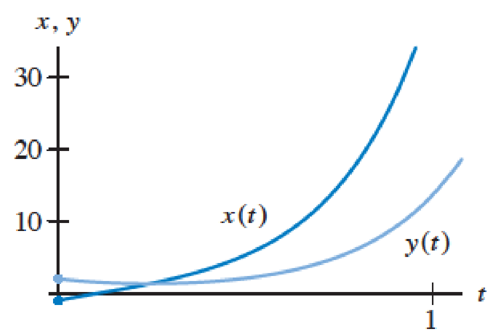
The initial condition $B = (3, 1)$ lies on the line $x = 3y$. Therefore, it corresponds to a straight-line solution, and the formula is $e^{4t}(3, 1)$.



The solution curve that corresponds to the initial condition $C = (0, -1)$ enters the third quadrant and eventually approaches line $x = 3y$. From the phase plane, we see that $x(t)$ is decreasing for all $t > 0$. We also see that $y(t)$ increases initially, reaches a negative maximum value, and then decreases in an exponential fashion. Since the solution curve crosses the line $y = x$, we know that these two graphs cross. By examining the line where $dy/dt = 0$, we see that these two graphs cross at precisely the same time as $y(t)$ attains its maximum value.



The solution curve that corresponds to the initial condition $D = (-1, 2)$ moves from the second quadrant into the first quadrant and eventually approaches the line $x = 3y$. From the phase plane, we see that $x(t)$ is increasing for all $t > 0$. We also see that $y(t)$ decreases initially, reaches a positive minimum value, and then increases in an exponential fashion. Since this solution curve crosses the line $y = x$, we know that these two graphs cross. By examining the line for which $dy/dt = 0$, we see that these two graphs cross at precisely the same time as $y(t)$ attains its minimum value.



1. Using Euler's formula, we can write the complex-valued solution $\mathbf{Y}_c(t)$ as

$$\begin{aligned}
\mathbf{Y}_c(t) &= e^{(1+3i)t}\begin{pmatrix} 2+i \\ 1 \end{pmatrix} \\
&= e^t e^{3it}\begin{pmatrix} 2+i \\ 1 \end{pmatrix} \\
&= e^t(\cos 3t + i\sin 3t)\begin{pmatrix} 2+i \\ 1 \end{pmatrix} \\
&= e^t\begin{pmatrix} 2\cos 3t - \sin 3t \\ \cos 3t \end{pmatrix} + ie^t\begin{pmatrix} 2\sin 3t + \cos 3t \\ \sin 3t \end{pmatrix}.
\end{aligned}$$

Hence, we have

$$\mathbf{Y}_{\text{re}}(t) = e^t\begin{pmatrix} 2\cos 3t - \sin 3t \\ \cos 3t \end{pmatrix} \quad \text{and} \quad \mathbf{Y}_{\text{im}}(t) = e^t\begin{pmatrix} \cos 3t + 2\sin 3t \\ \sin 3t \end{pmatrix}.$$

The general solution is

$$\mathbf{Y}(t) = k_1 e^t\begin{pmatrix} 2\cos 3t - \sin 3t \\ \cos 3t \end{pmatrix} + k_2 e^t\begin{pmatrix} \cos 3t + 2\sin 3t \\ \sin 3t \end{pmatrix}.$$

2. The complex solution is

$$\mathbf{Y}_c(t) = e^{(-2+5i)t} \begin{pmatrix} 1 \\ 4-3i \end{pmatrix},$$

so we can use Euler's formula to write

$$\begin{aligned} \mathbf{Y}_c(t) &= e^{(-2+5i)t} \begin{pmatrix} 1 \\ 4-3i \end{pmatrix} \\ &= e^{-2t} e^{5it} \begin{pmatrix} 1 \\ 4-3i \end{pmatrix} \\ &= e^{-2t}(\cos 5t + i \sin 5t) \begin{pmatrix} 1 \\ 4-3i \end{pmatrix} \\ &= e^{-2t} \begin{pmatrix} \cos 5t \\ 4\cos 5t + 3\sin 5t \end{pmatrix} + i e^{-2t} \begin{pmatrix} \sin 5t \\ 4\sin 5t - 3\cos 5t \end{pmatrix}. \end{aligned}$$

Hence, we have

$$\mathbf{Y}_{\mathrm{re}}(t) = e^{-2t} \begin{pmatrix} \cos 5t \\ 4\cos 5t + 3\sin 5t \end{pmatrix} \quad \text{and} \quad \mathbf{Y}_{\mathrm{im}}(t) = e^{-2t} \begin{pmatrix} \sin 5t \\ 4\sin 5t - 3\cos 5t \end{pmatrix}.$$

The general solution is

$$\mathbf{Y}(t) = k_1 e^{-2t} \begin{pmatrix} \cos 5t \\ 4\cos 5t + 3\sin 5t \end{pmatrix} + k_2 e^{-2t} \begin{pmatrix} \sin 5t \\ 4\sin 5t - 3\cos 5t \end{pmatrix}.$$

3. **(a)** The characteristic equation is

$$(-\lambda)^2 + 4 = \lambda^2 + 4 = 0,$$

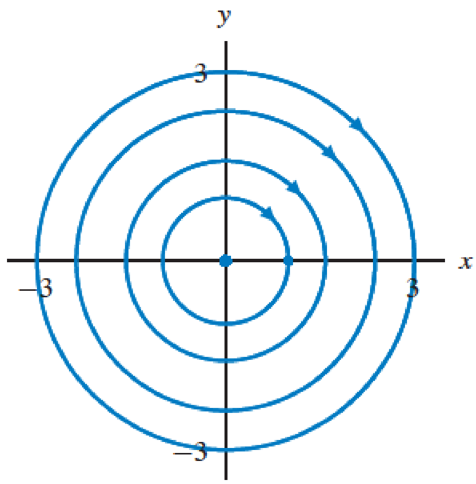
and the eigenvalues are $\lambda = \pm 2i$.

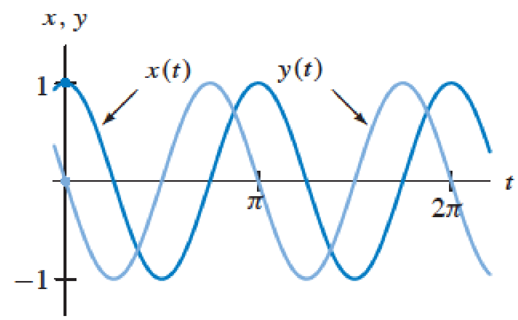
(b) Since the real part of the eigenvalues are 0, the origin is a center.

(c) Since $\lambda = \pm 2i$, the natural period is $2\pi/2 = \pi$, and the natural frequency is $1/\pi$.

(d) At $(1, 0)$, the tangent vector is $(-2, 0)$. Therefore, the direction of oscillation is clockwise.

(e) According to the phase plane, $x(t)$ and $y(t)$ are periodic with period π. At the initial condition $(1, 0)$, both $x(t)$ and $y(t)$ are initially decreasing.





4. **(a)** The characteristic equation is

$$(2-\lambda)(6-\lambda)+8=\lambda^2-8\lambda+20,$$

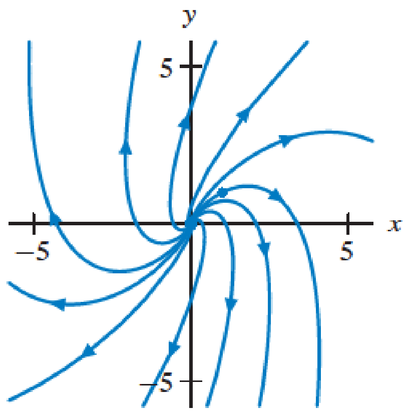
and the eigenvalues are $\lambda = 4 \pm 2i$.

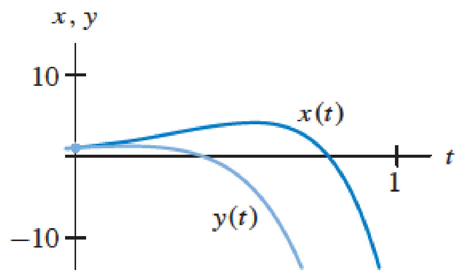
(b) Since the real part of the eigenvalues is positive, the origin is a spiral source.

(c) Since $\lambda = 4 \pm 2i$, the natural period is $2\pi/2 = \pi$, and the natural frequency is $1/\pi$.

(d) At the point $(1, 0)$, the tangent vector is $(2, -4)$. Therefore, the solution curves spiral around the origin in a clockwise fashion.

(e) Since $d\mathbf{Y}/dt = (4, 2)$ at $\mathbf{Y}_0 = (1, 1)$, both $x(t)$ and $y(t)$ increase initially. The distance between successive zeros is π, and the amplitudes of both $x(t)$ and $y(t)$ are increasing.





16. The characteristic polynomial is

$$(a-\lambda)(a-\lambda)+b^2=\lambda^2-2a\lambda+(a^2+b^2),$$

so the eigenvalues are

$$\lambda = \frac{2a \pm \sqrt{4a^2-4(a^2+b^2)}}{2} = a \pm \frac{\sqrt{-4b^2}}{2} = a \pm \sqrt{-b^2}.$$

Since $b^2 > 0$, the eigenvalues are complex. In fact, they are $a \pm bi$.

23. **(a)** The corresponding first-order system is

$$\frac{dy}{dt} = v$$
$$\frac{dv}{dt} = -qy - pv.$$

(b) The characteristic polynomial is

$$(-\lambda)(-p - \lambda) + q = \lambda^2 + p\lambda + q,$$

so the eigenvalues are $\lambda = (-p \pm \sqrt{p^2 - 4q}\,)/2$. Hence, the eigenvalues are complex if and only if $p^2 < 4q$. Note that q must be positive for this condition to be satisfied.

(c) In order to have a spiral sink, we must have $p^2 < 4q$ (to make the eigenvalues complex) and $p > 0$ (to make the real part of the eigenvalues negative). In other words, the origin is a spiral sink if and only if $q > 0$ and $0 < p < 2\sqrt{q}$. The origin is a center if and only if $q > 0$ and $p = 0$. Finally, the origin is a spiral source if and only if $q > 0$ and $-2\sqrt{q} < p < 0$.

(d) The vector field at $(1, 0)$ is $(0, -q)$. Hence, if $q > 0$, then the vector field points down along the entire y-axis, and the solution curves spiral about the origin in a clockwise fashion. Note that q must be positive for the eigenvalues to be complex, so the solution curves always spiral about the origin in a clockwise fashion as long as the eigenvalues are complex.

3. **(a)** The characteristic equation is

$$(-2-\lambda)(-4-\lambda)+1=(\lambda+3)^2=0,$$

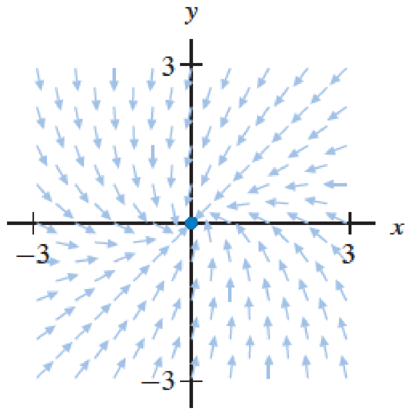
and the eigenvalue is $\lambda=-3$.

(b) To find an eigenvector, we solve the simultaneous equations

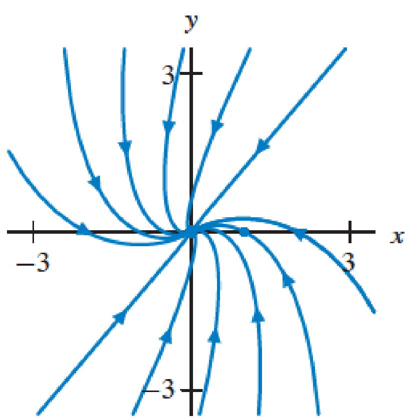
$$\begin{cases} -2x-y=-3x \\ x-4y=-3y. \end{cases}$$

Then, $y=x$, and one eigenvector is $(1,1)$.

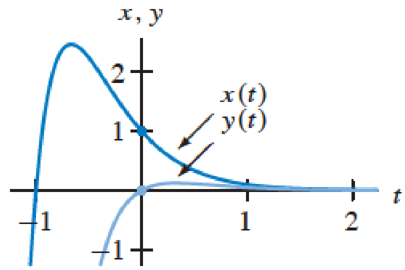
(c) Note the straight-line solutions along the line $y=x$.



(d) Since the eigenvalue is negative, any solution on the line $y=x$ tends toward the origin along $y=x$ as t increases. According to the direction field, every solution tends to the origin as t increases. The solutions with initial conditions that lie in the half-plane $y>x$ eventually approach the origin tangent to the half-line $y=x$ with $y<0$. Similarly, the solutions with initial conditions that lie in the half-plane $y<x$ eventually approach the origin tangent to the line $y=x$ with $y>0$.



(e) At the point $\mathbf{Y}_0 = (1, 0)$, $d\mathbf{Y}/dt = (-2, 1)$. Therefore, $x(t)$ initially decreases and $y(t)$ initially increases. The solution eventually approaches the origin tangent to the line $y = x$. Since the solution curve never crosses the line $y = x$, the graphs of $x(t)$ and $y(t)$ do not cross.



4. **(a)** The characteristic polynomial is

$$(-\lambda)(-2-\lambda) + 1 = \lambda^2 + 2\lambda + 1 = (\lambda + 1)^2,$$

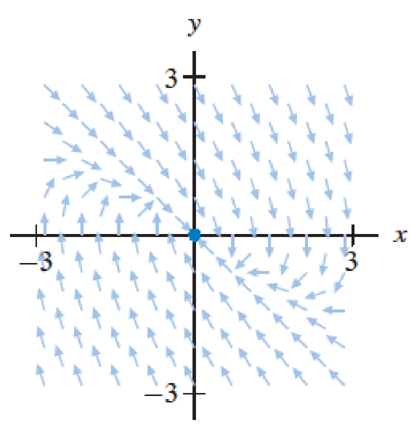
so there is only one eigenvalue, $\lambda = -1$.

(b) To find an eigenvalue we solve

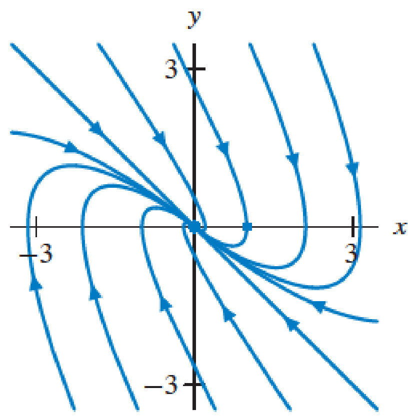
$$\begin{cases} y = -x \\ -x - 2y = -y. \end{cases}$$

These equations both simplify to $y = -x$, so $(1, -1)$ is one eigenvector.

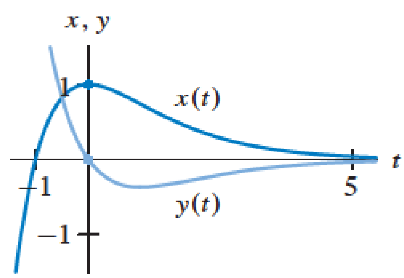
(c) Note the straight-line solutions along the line $y = -x$.



(d) Since the eigenvalue is negative, all solutions approach the origin as t increases. Solutions with initial conditions on the line $y = -x$ approach the origin along $y = -x$. Solutions with initial conditions that lie in the half-plane $y > -x$ approach the origin tangent to the half-line $y = -x$ with $y < 0$. Solutions with initial conditions that lie in the half-plane $y < -x$ approach the origin tangent to the half-line $y = -x$ with $y > 0$.



(e) At the point $\mathbf{Y}_0 = (1, 0)$, $d\mathbf{Y}/dt = (0, -1)$. Therefore, $x(t)$ assumes a maximum at $t = 0$ and then decreases toward 0. Also, $y(t)$ becomes negative. Then, it assumes a (negative) minimum, and finally it is asymptotic to 0 without crossing $y = 0$.



9. **(a)** By solving the quadratic equation, we obtain

$$\lambda = \frac{-\alpha \pm \sqrt{\alpha^2 - 4\beta}}{2}.$$

Therefore, for the quadratic to have a double root, we must have

$$\alpha^2 - 4\beta = 0.$$

(b) If zero is a root, we set $\lambda = 0$ in $\lambda^2 + \alpha\lambda + \beta = 0$, and we obtain $\beta = 0$.

10. **(a)** To compute the limit of $te^{\lambda t}$ as $t \to \infty$ if $\lambda > 0$, we note that both t and $e^{\lambda t}$ go to infinity as t goes to infinity. So $te^{\lambda t}$ blows up as t tends to infinity, and the limit does not exist.

(b) To compute the limit of $te^{\lambda t}$ as $t \to \infty$ if $\lambda < 0$, we write

$$\lim_{t\to\infty} te^{\lambda t} = \lim_{t\to\infty} \frac{t}{e^{-\lambda t}} = \lim_{t\to\infty} \frac{1}{-\lambda e^{-\lambda t}}$$

where the last equality follows from L'Hôpital's Rule. Because $e^{-\lambda t}$ tends to infinity as $t \to \infty$ $(-\lambda > 0)$, the fraction tends to 0.

12. The characteristic polynomial of **A** is

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \lambda^2 - (a+d)\lambda + (ad - bc) = \lambda^2 - \mathrm{tr}(\mathbf{A})\lambda + \det(\mathbf{A})$$

(see Section 3.2). A quadratic polynomial has only one root if and only if its discriminant is 0. In this case, the discriminant of $\det(\mathbf{A} - \lambda\mathbf{I})$ is $\mathrm{tr}(\mathbf{A})^2 - 4\det(\mathbf{A})$.

18. **(a)** The characteristic equation is

$$(2-\lambda)(6-\lambda) - 12 = \lambda^2 - 8\lambda = 0.$$

Therefore, the eigenvalues are $\lambda = 0$ and $\lambda = 8$.

(b) To find the eigenvectors $\mathbf{V}_1$ associated to the eigenvalue $\lambda = 0$, we must solve $\mathbf{A}\mathbf{V}_1 = 0\mathbf{V}_1 = 0$ where $\mathbf{A}$ is the matrix that defines this linear system. (Note that this is the same calculation we do if we want to locate the equilibrium points.) We get

$$\begin{cases} 2x_1 + 4y_1 = 0 \\ 3x_1 + 6y_1 = 0, \end{cases}$$

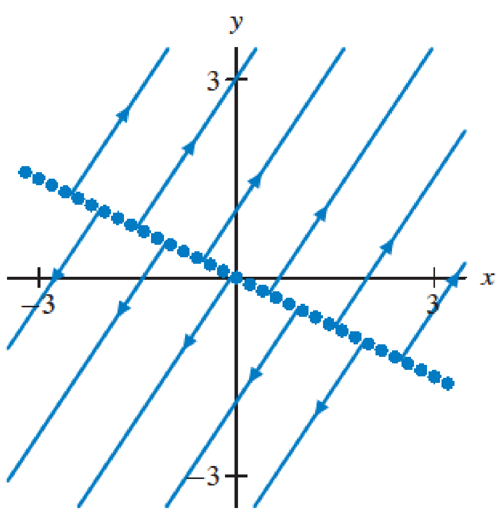
where $\mathbf{V}_1 = (x_1, y_1)$. Hence, the eigenvectors associated to $\lambda = 0$ (as well as the equilibrium points) must satisfy the equation $x_1 + 2y_1 = 0$.

To find the eigenvectors $\mathbf{V}_2$ associated to the eigenvalue $\lambda = 8$, we must solve $\mathbf{A}\mathbf{V}_2 = 8\mathbf{V}_2$. We get

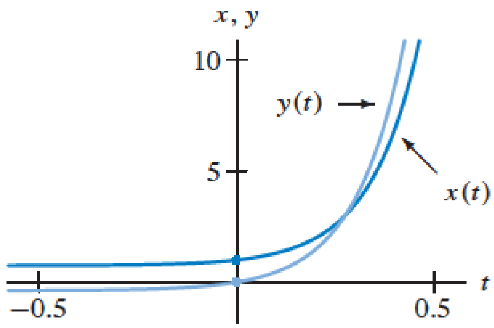
$$\begin{cases} 2x_2 + 4y_2 = 8x_2 \\ 3x_2 + 6y_2 = 8y_2, \end{cases}$$

where $\mathbf{V}_2 = (x_2, y_2)$. Hence, the eigenvectors associated to $\lambda = 8$ must satisfy $2y_2 = 3x_2$.

(c) The equation $x_1 + 2y_1 = 0$ specifies a line of equilibrium points. Since the other eigenvalue is positive, solution curves not corresponding to equilibria move away from this line as t increases.



(d) As t increases, both $x(t)$ and $y(t)$ increase exponentially. As t decreases, both x and y approach constants that are determined by the line of equilibrium points.



(e) To form the general solution, we must pick one eigenvector for each eigenvalue. Using part (b), we pick $\mathbf{V}_1 = (-2, 1)$, and $\mathbf{V}_2 = (2, 3)$. We obtain the general solution

$$\mathbf{Y}(t) = k_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} + k_2 e^{8t} \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

(f) To determine the solution whose initial condition is $(1, 0)$, we let $t = 0$ in the general solution and obtain the equations

$$k_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} + k_2 \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

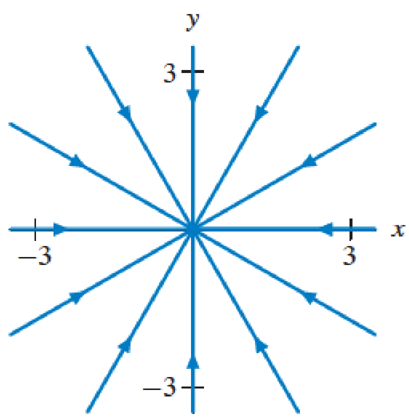
Therefore, $k_1 = -3/8$ and $k_2 = 1/8$. The particular solution is

$$\mathbf{Y}(t) = \begin{pmatrix} \frac{3}{4} + \frac{1}{4} e^{8t} \\ -\frac{3}{8} + \frac{3}{8} e^{8t} \end{pmatrix}.$$

23. **(a)** The characteristic polynomial is $(a-\lambda)(d-\lambda)$, so the eigenvalues are a and d.

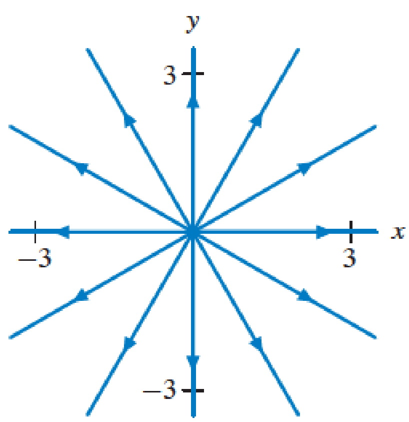
(b) If $a \neq d$, the lines of eigenvectors for a and d are the x- and y-axes respectively.

(c) If $a = d < 0$, every nonzero vector is an eigenvector (see Exercise 14), and all the vectors point toward the origin. Hence, every solution curve is asymptotic to the origin along a straight line.



The general solution is $\mathbf{Y}(t) = e^{at}\mathbf{Y}_0$, where $\mathbf{Y}_0$ is the initial condition.

(d) The only difference between this case and part (c) is that the arrows in the vector field are reversed. Every solution tends away from the origin along a straight line.



Again the general solution is $\mathbf{Y}(t) = e^{at}\mathbf{Y}_0$, where $\mathbf{Y}_0$ is the initial condition.