1. **(a)** The function $g_a(t) = 1$ precisely when $u_a(t) = 0$, and $g_a(t) = 0$ precisely when $u_a(t) = 1$, so

$$g_a(t) = 1 - u_a(t).$$

(b) We can compute the Laplace transform of $g_a(t)$ from the definition

$$\mathcal{L}[g_a] = \int_0^a 1e^{-st}\,dt = -\frac{e^{-as}}{s} + \frac{e^{-0s}}{s} = \frac{1}{s} - \frac{e^{-as}}{s}.$$

Alternately, we can use the table

$$\mathcal{L}[g_a] = \mathcal{L}[1 - u_a(t)] = \frac{1}{s} - \frac{e^{-as}}{s}.$$

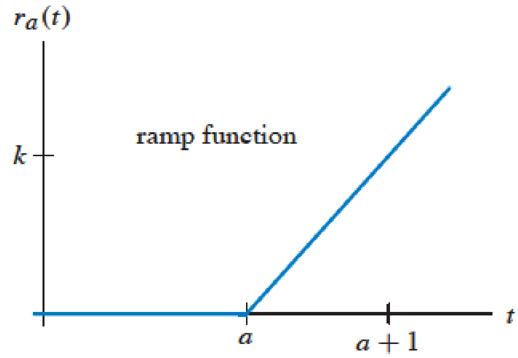
2. **(a)** We have $r_a(t) = u_a(t)y(t-a)$, where $y(t) = kt$. Now

$$\mathcal{L}[y(t)] = k\mathcal{L}[t] = \frac{k}{s^2},$$

so using the rules of Laplace transform,

$$\mathcal{L}[r_a(t)] = \mathcal{L}[u_a(t)y(t-a)] = \frac{k}{s^2}e^{-as}.$$

(b)



4. We have

$$\mathcal{L}[e^{3t}] = \frac{1}{s-3},$$

so using the rule

$$\mathcal{L}[u_a(t)y(t-a)] = e^{-as}\mathcal{L}[y(t)],$$

we determine that

$$\mathcal{L}[u_2(t)e^{3(t-2)}] = \frac{e^{-2s}}{s-3}.$$

The desired function is $u_2(t)e^{3(t-2)}$.

8. Taking the Laplace transform of both sides of the equation gives us

$$\mathcal{L}\left[\frac{dy}{dt}\right] = \mathcal{L}[u_2(t)],$$

so

$$s\mathcal{L}[y] - y(0) = \frac{e^{-2s}}{s}.$$

Substituting the initial condition yields

$$s\mathcal{L}[y] - 3 = \frac{e^{-2s}}{s},$$

so that

$$\mathcal{L}[y] = \frac{e^{-2s}}{s^2} + \frac{3}{s}.$$

By taking the inverse of the Laplace transform, we get

$$y(t) = u_2(t)(t-2) + 3.$$

To check our answer, we compute

$$\frac{dy}{dt} = \frac{du_2}{dt}(t-2) + u_2(t),$$

and since $du_2/dt = 0$ except at $t = 2$ (where it is undefined),

$$\frac{dy}{dt} = u_2(t).$$

Hence, our $y(t)$ satisfies the differential equation except when $t = 2$. (We cannot expect $y(t)$ to satisfy the differential equation at $t = 2$ because the differential equation is not continuous there.) Note that $y(t)$ also satisfies the initial condition $y(0) = 3$.

15. Taking the Laplace transform of both sides of the equation, we have

$$\mathcal{L}\left[\frac{dy}{dt}\right] = -\mathcal{L}[y] + \mathcal{L}[u_a(t)],$$

which is equivalent to

$$s\mathcal{L}[y] - y(0) = -\mathcal{L}[y] + \frac{e^{-as}}{s}.$$

Solving for $\mathcal{L}[y]$ yields

$$\mathcal{L}[y] = \frac{e^{-as}}{s(s+1)} + \frac{y(0)}{s+1}.$$

Using the partial fractions decomposition

$$\frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1},$$

we get

$$\mathcal{L}[y] = \frac{e^{-as}}{s} - \frac{e^{-as}}{s+1} + \frac{y(0)}{s+1}.$$

Taking the inverse Laplace transform, we obtain

$$\begin{aligned} y(t) &= u_a(t) - u_a(t)e^{-(t-a)} + y(0)e^{-t} \\ &= u_a(t)\left(1 - e^{-(t-a)}\right) + y(0)e^{-t}. \end{aligned}$$

To check our answer, we compute

$$\frac{dy}{dt} = \frac{du_a}{dt}\left(1 - e^{-(t-a)}\right) + u_a(t)e^{-(t-a)} - y(0)e^{-t}$$

and since $du_a/dt = 0$ except at $t = a$ (where it is undefined),

$$\begin{aligned} \frac{dy}{dt} + y &= u_a(t)e^{-(t-a)} - y(0)e^{-t} + u_a(t)\left(1 - e^{-(t-a)}\right) + y(0)e^{-t} \\ &= u_a(t). \end{aligned}$$

Hence, our $y(t)$ satisfies the differential equation except when $t = a$. (We cannot expect $y(t)$ to satisfy the differential equation at $t = a$ because the differential equation is not continuous there.)

16. We can write $\mathcal{L}[f]$ as the sum of two integrals, that is,

$$\mathcal{L}[f] = \int_0^\infty f(t)\, e^{-st}\, dt = \int_0^T f(t)\, e^{-st}\, dt + \int_T^\infty f(t)\, e^{-st}\, dt.$$

Next, we use the substitution $u = t - T$ on the second integral. Note that $t = u + T$. We get

$$\int_T^\infty f(t)\, e^{-st}\, dt = \int_0^\infty f(u+T)\, e^{-s(u+T)}\, du.$$

Since f is periodic with period T, we can rewrite the last integral as

$$e^{-Ts} \int_0^\infty f(u)\, e^{-su}\, du,$$

which is just $e^{-Ts}\mathcal{L}[f]$. Hence,

$$\mathcal{L}[f] = \int_0^T f(t)\, e^{-st}\, dt + e^{-Ts}\mathcal{L}[f].$$

We have

$$(1 - e^{-Ts})\mathcal{L}[f] = \int_0^T f(t)\, e^{-st}\, dt.$$

Consequently,

$$\mathcal{L}[f] = \frac{1}{1 - e^{-Ts}} \int_0^T f(t)\, e^{-st}\, dt.$$

18. From the formula in Exercise 16, we see that we need only compute the integral $\int_0^1 te^{-st}\, dt$. Using integration by parts (as in Exercise 2 of Section 6.1), we get

$$\begin{aligned}\mathcal{L}[z] &= \frac{1}{1 - e^{-s}} \left(\frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-s}}{s} \right) \\ &= \frac{1}{s^2} - \frac{e^{-s}}{s(1 - e^{-s})}.\end{aligned}$$

5. Using the formula

$$\mathcal{L}\left[\frac{d^2y}{dt^2}\right] = s^2\mathcal{L}[y] - y'(0) - sy(0),$$

and the linearity of the Laplace transform, we get that

$$s^2\mathcal{L}[y] - y'(0) - sy(0) + \omega^2\mathcal{L}[y] = 0.$$

Substituting the initial conditions and solving for $\mathcal{L}[y]$ gives

$$\mathcal{L}[y] = \frac{s}{s^2 + \omega^2}.$$

6. Since

$$\mathcal{L}[\cos \omega t] = \frac{s}{s^2 + \omega^2},$$

we can compute that

$$\frac{d}{d\omega}\mathcal{L}[\cos \omega t] = \frac{-s(2\omega)}{(s^2+\omega^2)^2} = \frac{-2\omega s}{(s^2+\omega^2)^2},$$

but

$$\frac{d}{d\omega}\mathcal{L}[\cos \omega t] = \mathcal{L}\left[\frac{d}{d\omega}\cos \omega t\right] = \mathcal{L}[-t \sin \omega t].$$

We can bring the derivative with respect to ω inside the Laplace transform because the Laplace transform is an integral with respect to t, that is,

$$\frac{d}{d\omega}\mathcal{L}[\cos \omega t] = \frac{d}{d\omega}\int_0^\infty \cos \omega t\, e^{-st}\, dt = \int_0^\infty \frac{d}{d\omega}\left(\cos \omega t\, e^{-st}\right)\, dt.$$

Canceling the minus signs on left and right gives

$$\mathcal{L}[t \sin \omega t] = \frac{2\omega s}{(s^2+\omega^2)^2}.$$

8. We need to compute

$$\mathcal{L}[te^{at}] = \int_0^\infty te^{at}e^{-st}\, dt.$$

We can do this using the hint, by differentiating $\mathcal{L}[e^{at}]$ with respect to a. Another method is to write

$$\mathcal{L}[te^{at}] = \int_0^\infty te^{at}e^{-st}\, dt = \int_0^\infty te^{-(s-a)t}\, dt = \int_0^\infty te^{-rt}\, dt$$

where $r = s - a$. The last integral is the Laplace transform of t using r as the new independent variable. Hence, from the table we have

$$\int_0^\infty te^{-rt}\, dt = \frac{1}{r^2}.$$

Substituting back $r = s - a$ we have

$$\mathcal{L}[te^{at}] = \frac{1}{(s-a)^2}.$$

15. In Exercise 11, we completed the square and obtained $s^2 + 2s + 10 = (s+1)^2 + 3^2$, so

$$\begin{aligned}\mathcal{L}^{-1}\left[\frac{1}{s^2+2s+10}\right] &= \mathcal{L}^{-1}\left[\frac{1}{(s+1)^2+3^2}\right] \\ &= \frac{1}{3}\mathcal{L}^{-1}\left[\frac{3}{(s+1)^2+3^2}\right] \\ &= \frac{1}{3}e^{-t}\sin 3t.\end{aligned}$$

18. In Exercise 14, we completed the square and obtained $s^2 + 6s + 10 = (s+3)^2 + 1^2$, so

$$\frac{s+1}{s^2+6s+10} = \frac{s+1}{(s+3)^2+1^2}.$$

We want to put this fraction in the right form so that we can use the formulas for $\mathcal{L}[e^{at}\cos\omega t]$ and $\mathcal{L}[e^{at}\sin\omega t]$. We see that

$$\frac{s+1}{(s+3)^2+1^2} = \frac{s+3}{(s+3)^2+1^2} - \frac{2}{(s+3)^2+1^2}.$$

So

$$\mathcal{L}^{-1}\left[\frac{s+1}{s^2+6s+10}\right] = e^{-3t}\cos t - 2e^{-3t}\sin t.$$

27. **(a)** Taking the Laplace transform of both sides of the equation, we obtain

$$\mathcal{L}\left[\frac{d^2y}{dt^2}\right] + 4\mathcal{L}[y] = \frac{8}{s},$$

and using the fact that $\mathcal{L}[d^2y/dt^2] = s^2\mathcal{L}[y] - sy(0) - y'(0)$, we have

$$(s^2+4)\mathcal{L}[y] - sy(0) - y'(0) = \frac{8}{s}.$$

(b) Substituting the initial conditions yields

$$(s^2+4)\mathcal{L}[y] - 11s - 5 = \frac{8}{s},$$

and solving for $\mathcal{L}[y]$ we get

$$\mathcal{L}[y] = \frac{11s+5}{s^2+4} + \frac{8}{s(s^2+4)}.$$

The partial fractions decomposition of $8/(s(s^2+4))$ is

$$\frac{8}{s(s^2+4)} = \frac{A}{s} + \frac{Bs+C}{s^2+4}.$$

Putting the right-hand side over a common denominator gives us

$$(A+B)s^2 + Cs + 4A = 8,$$

and consequently, $A = 2$, $B = -2$, and $C = 0$. In other words,

$$\frac{8}{s(s^2+4)} = \frac{2}{s} + \frac{-2s}{s^2+4}.$$

We obtain

$$\mathcal{L}[y] = \frac{2}{s} + \frac{9s+5}{s^2+4}.$$

(c) To take the inverse Laplace transform, we rewrite $\mathcal{L}[y]$ in the form

$$\mathcal{L}[y] = \frac{2}{s} + 9\left(\frac{s}{s^2+4}\right) + \frac{5}{2}\left(\frac{2}{s^2+4}\right).$$

Therefore, $y(t) = 2 + 9\cos 2t + \frac{5}{2}\sin 2t$.

28. **(a)** Taking the Laplace transform of both sides of the equation, we obtain

$$\mathcal{L}\left[\frac{d^2y}{dt^2}\right] - \mathcal{L}[y] = \frac{1}{s-2},$$

and using the fact that $\mathcal{L}[d^2y/dt^2] = s^2\mathcal{L}[y] - sy(0) - y'(0)$, we have

$$(s^2-1)\mathcal{L}[y] - sy(0) - y'(0) = \frac{1}{s-2}.$$

(b) Substituting the initial conditions yields

$$(s^2-1)\mathcal{L}[y] - s + 1 = \frac{1}{s-2},$$

and solving for $\mathcal{L}[y]$ we get

$$\mathcal{L}[y] = \frac{1}{s+1} + \frac{1}{(s-2)(s^2-1)}.$$

Using the partial fractions decomposition

$$\frac{1}{(s-2)(s^2-1)} = \frac{\frac{1}{3}}{s-2} + \frac{-\frac{1}{2}}{s-1} + \frac{\frac{1}{6}}{s+1},$$

we obtain

$$\mathcal{L}[y] = \frac{\frac{1}{3}}{s-2} + \frac{-\frac{1}{2}}{s-1} + \frac{\frac{7}{6}}{s+1}.$$

(c) Taking the inverse Laplace transform, we have

$$y(t) = \tfrac{1}{3}e^{2t} - \tfrac{1}{2}e^{t} + \tfrac{7}{6}e^{-t}.$$

1. This is the $\frac{0}{0}$ case of L'Hôpital's Rule. Differentiating numerator and denominator with respect to Δt, we obtain

$$\frac{se^{s\Delta t} - (-s)e^{-s\Delta t}}{2},$$

which simplifies to

$$\frac{s(e^{s\Delta t} + e^{-s\Delta t})}{2}.$$

Since both $e^{s\Delta t}$ and $e^{-s\Delta t}$ tend to 1 as $\Delta t \to 0$, the desired limit is s.

2. Taking Laplace transforms of both sides and applying the rules yields

$$s^2\mathcal{L}[y] - sy(0) - y'(0) + 3\mathcal{L}[y] = 5\mathcal{L}[\delta_2].$$

Simplifying, using the initial conditions, and the fact that $\mathcal{L}[\delta_2] = e^{-2s}$, we get

$$(s^2 + 3)\mathcal{L}[y] = 5e^{-2s}.$$

Hence,

$$\mathcal{L}[y] = 5\frac{e^{-2s}}{s^2 + 3}.$$

This can be written as

$$\mathcal{L}[y] = \frac{5}{\sqrt{3}}e^{-2s}\frac{\sqrt{3}}{s^2 + 3},$$

which yields

$$y(t) = \frac{5}{\sqrt{3}}u_2(t)\sin\left(\sqrt{3}(t - 2)\right).$$

6. **(a)** The characteristic polynomial of the unforced oscillator is $\lambda^2 + 2\lambda + 3$, and the eigenvalues are $\lambda = -1 \pm \sqrt{2}\,i$. Hence, the natural period is $\sqrt{2}\,\pi$ and the damping causes the solutions of the unforced equation to tend to zero like e^{-t}. At $t = 4$, the system is given a jolt, so the solution rises. After $t = 4$, the equation is unforced, so the solution again tends to zero as e^{-t}.

(b) Taking Laplace transforms of both sides of the equation, we have

$$s^2\mathcal{L}[y] - sy(0) - y'(0) + 2s\mathcal{L}[y] - 2y(0) + 3\mathcal{L}[y] = \mathcal{L}[\delta_4].$$

Plugging in the initial conditions and solving for $\mathcal{L}[y]$ gives us

$$\mathcal{L}[y] = \frac{s+2}{s^2+2s+3} + \frac{e^{-4s}}{s^2+2s+3}.$$

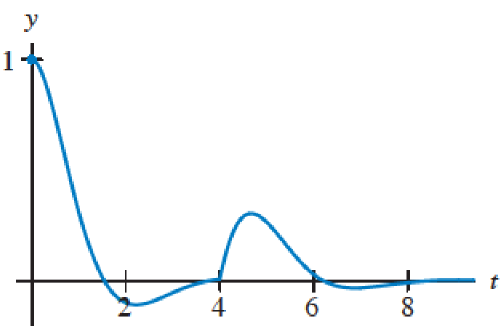
If we complete the square for the polynomial $s^2 + 2s + 3$, we get $s^2 + 2s + 3 = (s+1)^2 + 2$, so

$$\mathcal{L}[y] = \frac{s+1}{(s+1)^2+2} + \frac{1}{\sqrt{2}}\frac{\sqrt{2}}{(s+1)^2+2} + \frac{1}{\sqrt{2}}e^{-4s}\frac{\sqrt{2}}{(s+1)^2+2}.$$

Therefore,

$$y(t) = e^{-t}\cos\sqrt{2}\,t + \frac{1}{\sqrt{2}}e^{-t}\sin\sqrt{2}\,t + \frac{1}{\sqrt{2}}u_4(t)e^{-(t-4)}\sin(\sqrt{2}(t-4)).$$

(c)



Note that the solution goes through about 3/4 of a natural period before the application of the delta function. The delta function forcing causes the second maximum of the solution to be much higher than it would have been without the forcing, but the long term effect is small because the damping is fairly large.

7. **(a)** From the table

$$\mathcal{L}[\delta_a] = e^{-as}$$

$$s\mathcal{L}[u_a] - u_a(0) = s\frac{e^{-as}}{s} - 0 = e^{-as}.$$

(b) The formula for the Laplace transform of a derivative is

$$\mathcal{L}\left[\frac{dy}{dt}\right] = s\mathcal{L}[y] - y(0)$$

and this is exactly the relationship between the Laplace transforms of $u_a(t)$ and $\delta_a(t)$. Hence, it is tempting to think of the Dirac delta function as the derivative of the Heaviside function.

(c) We can think of the Heaviside function $u_a(t)$ as a limit of piecewise linear functions equal to zero for t less than $a - \Delta t$, equal to one for t greater than $a + \Delta t$ and a straight line for t between $a - \Delta t$ and $a + \Delta t$. The derivative of this function is precisely the function $g_{\Delta t}$ used to define the Dirac delta function. This is still just an informal relationship until we specify in what sense we are taking the limit.