1. We have

$$\begin{aligned}\mathcal{L}[3] &= \int_0^\infty 3e^{-st}\,dt \\ &= \lim_{b\to\infty}\int_0^b 3e^{-st}\,dt \\ &= \lim_{b\to\infty}\left(\frac{-3}{s}e^{-st}\Big|_0^b\right) \\ &= \lim_{b\to\infty} -\frac{3}{s}\left(e^{-sb}-e^0\right) \\ &= \frac{3}{s} \quad \text{if } s>0,\end{aligned}$$

since $\lim_{b\to\infty} e^{-sb} = \lim_{b\to\infty} 1/e^{sb} = 0$ if $s > 0$.

2. We have

$$\mathcal{L}[t] = \int_0^\infty te^{-st}\,dt = \lim_{b\to\infty}\int_0^b te^{-st}\,dt.$$

To evaluate the integral we use integration by parts with $u = t$ and $dv = e^{-st}dt$. Then $du = dt$ and $v = -e^{-st}/s$. Thus

$$\begin{aligned}\lim_{b\to\infty}\int_0^b te^{-st}\,dt &= \lim_{b\to\infty}\left(-\frac{te^{-st}}{s}\Big|_0^b - \int_0^b -\frac{e^{-st}}{s}\,dt\right) \\ &= \lim_{b\to\infty}\left(-\frac{be^{-sb}}{s} - \frac{e^{-st}}{s^2}\Big|_0^b\right) \\ &= \lim_{b\to\infty}\left(-\frac{be^{-sb}}{s} - \frac{e^{-sb}}{s^2} + \frac{e^0}{s^2}\right) \\ &= \frac{1}{s^2}\end{aligned}$$

since

$$\lim_{b\to\infty} -\frac{be^{-sb}}{s} = \lim_{b\to\infty}\frac{-b}{se^{sb}} = \lim_{b\to\infty}\frac{-1}{s^2e^{sb}} = 0$$

by L'Hôpital's Rule if $s > 0$.

3. We use the fact that $\mathcal{L}[df/dt] = s\mathcal{L}[f] - f(0)$. Letting $f(t) = t^2$ we have $f(0) = 0$ and

$$\mathcal{L}[2t] = s\mathcal{L}[t^2] - 0$$

or

$$2\mathcal{L}[t] = s\mathcal{L}[t^2]$$

using the fact that the Laplace transform is linear. Then since $\mathcal{L}[t] = 1/s^2$ (by the previous exercise), we have

$$\mathcal{L}[-5t^2] = -5\mathcal{L}[t^2] = -5\left(\frac{2\mathcal{L}[t]}{s}\right) = -\frac{10}{s^3}.$$

5. To show a rule by induction, we need two steps. First, we need to show the rule is true for $n = 1$. Then, we need to show that if the rule holds for n, then it holds for $n + 1$.

(a) $n = 1$. We need to show that $\mathcal{L}[t] = 1/s^2$.

$$\mathcal{L}[t] = \int_0^\infty te^{-st}\,dt.$$

Using integration by parts with $u = t$ and $dv = e^{-st}\,dt$, we find

$$\begin{aligned}
\mathcal{L}[t] &= \left.\frac{te^{-st}}{-s}\right|_0^\infty + \int_0^\infty \frac{e^{-st}}{s}\,dt \\
&= \lim_{b\to\infty}\left[\left.\frac{te^{-st}}{-s}\right|_0^b\right] + \int_0^\infty \frac{e^{-st}}{s}\,dt \\
&= \int_0^\infty \frac{e^{-st}}{s}\,dt \\
&= \left.-\frac{e^{-st}}{s^2}\right|_0^\infty \\
&= \frac{1}{s^2} \quad (s > 0).
\end{aligned}$$

(b) Now we assume that the rule holds for n, that is, that $\mathcal{L}[t^n] = n!/s^{n+1}$, and show it holds true for $n+1$, that is, $\mathcal{L}[t^{n+1}] = (n+1)!/s^{n+2}$. There are two different methods to do so:

(i)

$$\mathcal{L}[t^{n+1}] = \int_0^\infty t^{n+1}\, e^{-st}\, dt$$

Using integration by parts with $u = t^{n+1}$ and $dv = e^{-st}\, dt$, we find

$$\mathcal{L}[t^{n+1}] = -\frac{t^{n+1}e^{-st}}{s}\bigg|_0^\infty + \int_0^\infty \frac{n+1}{s} t^n\, e^{-st}\, dt.$$

Now,

$$\begin{aligned} -\frac{t^{n+1}e^{-st}}{s}\bigg|_0^\infty &= \lim_{b\to\infty}\left[-\frac{t^{n+1}e^{-st}}{s}\bigg|_0^b\right] \\ &= \lim_{b\to\infty}\frac{-b^{n+1}e^{-sb}}{s} + 0 \\ &= 0 \quad (s > 0). \end{aligned}$$

So

$$\begin{aligned} \mathcal{L}[t^{n+1}] &= \int_0^\infty \frac{n+1}{s} t^n\, e^{-st}\, dt \\ &= \frac{n+1}{s}\int_0^\infty t^n\, e^{-st}\, dt \\ &= \frac{n+1}{s}\mathcal{L}[t^n]. \end{aligned}$$

Since we assumed that $\mathcal{L}[t^n] = n!/s^{n+1}$, we get that

$$\mathcal{L}[t^{n+1}] = \frac{n+1}{s} \cdot \frac{n!}{s^{n+1}} = \frac{(n+1)!}{s^{n+2}}$$

which is what we wanted to show.

(ii) We use the fact that $\mathcal{L}[df/dt] = s\mathcal{L}[f] - f(0)$. Letting $f(t) = t^{n+1}$ we have $f(0) = 0$ and

$$\mathcal{L}[(n+1)t^n] = s\mathcal{L}[t^{n+1}] - 0$$

or

$$(n+1)\mathcal{L}[t^n] = s\mathcal{L}[t^{n+1}]$$

using the fact that the Laplace transform is linear. Since we assumed $\mathcal{L}[t^n] = n!/s^{n+1}$, we have

$$\mathcal{L}[t^{n+1}] = \frac{n+1}{s}\mathcal{L}[t^n] = \frac{n+1}{s} \cdot \frac{n!}{s^{n+1}} = \frac{(n+1)!}{s^{n+2}},$$

which is what we wanted to show.

7. Since we know that $\mathcal{L}[e^{at}] = 1/(s-a)$, we have $\mathcal{L}[e^{3t}] = 1/(s-3)$, and therefore,

$$\mathcal{L}^{-1}\left[\frac{1}{s-3}\right] = e^{3t}.$$

8. We see that

$$\frac{5}{3s} = \frac{5}{3} \cdot \frac{1}{s},$$

so

$$\mathcal{L}^{-1}\left[\frac{5}{3s}\right] = \frac{5}{3},$$

since $\mathcal{L}^{-1}[1/s] = 1$.

9. We see that

$$\frac{2}{3s+5} = \frac{2}{3} \cdot \frac{1}{s+5/3},$$

so

$$\mathcal{L}^{-1}\left[\frac{2}{3s+5}\right] = \frac{2}{3}e^{-\frac{5}{3}t}.$$

15. **(a)** We have

$$\mathcal{L}\left[\frac{dy}{dt}\right] = s\mathcal{L}[y] - y(0)$$

and

$$\mathcal{L}[-y + e^{-2t}] = \mathcal{L}[-y] + \mathcal{L}[e^{-2t}] = -\mathcal{L}[y] + \frac{1}{s+2}$$

using linearity of the Laplace transform and the formula $\mathcal{L}[e^{at}] = 1/(s-a)$ from the text.

(b) Substituting the initial condition yields

$$s\mathcal{L}[y] - 2 = -\mathcal{L}[y] + \frac{1}{s+2}$$

so that

$$(s+1)\mathcal{L}[y] = 2 + \frac{1}{s+2}$$

which gives

$$\mathcal{L}[y] = \frac{1}{(s+1)(s+2)} + \frac{2}{s+1} = \frac{2s+5}{(s+1)(s+2)}.$$

(c) Using the method of partial fractions,

$$\frac{2s+5}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}.$$

Putting the right-hand side over a common denominator gives $A(s+2) + B(s+1) = 2s+5$, which can be written as $(A+B)s + (2A+B) = 2s+5$. So we have $A+B = 2$, and $2A+B = 5$. Thus, $A = 3$ and $B = -1$, and

$$\mathcal{L}[y] = \frac{3}{s+1} - \frac{1}{s+2}.$$

Therefore, $y(t) = 3e^{-t} - e^{-2t}$ is the desired function.

(d) Since $y(0) = 3e^0 - e^0 = 2$, $y(t)$ satisfies the given initial condition. Also,

$$\frac{dy}{dt} = -3e^{-t} + 2e^{-2t}$$

and

$$-y + e^{-2t} = -3e^{-t} + e^{-2t} + e^{-2t} = -3e^{-t} + 2e^{-2t},$$

so our solution also satisfies the differential equation.

18. **(a)** Taking Laplace transforms of both sides of the equation and simplifying gives

$$\mathcal{L}\left[\frac{dy}{dt}\right] + 4\mathcal{L}[y] = \mathcal{L}[6]$$

so

$$s\mathcal{L}[y] - y(0) + 4\mathcal{L}[y] = \frac{6}{s}$$

and $y(0) = 0$ gives

$$s\mathcal{L}[y] + 4\mathcal{L}[y] = \frac{6}{s}.$$

(b) Solving for $\mathcal{L}[y]$ gives

$$\mathcal{L}[y] = \frac{6}{s(s+4)}.$$

(c) Using the method of partial fractions,

$$\frac{6}{s(s+4)} = \frac{A}{s} + \frac{B}{s+4}.$$

Putting the right-hand side over a common denominator gives $A(s+4) + Bs = 6$, which can be written as $(A+B)s + 4A = 6$. So, $A + B = 0$, and $4A = 6$. Hence, $A = 3/2$ and $B = -3/2$, and we have

$$\mathcal{L}[y] = \frac{3/2}{s} - \frac{3/2}{s+4}.$$

Thus,

$$y(t) = \frac{3}{2} - \frac{3}{2}e^{-4t}.$$

(d) To check, we compute

$$\frac{dy}{dt} + 4y = 6e^{-4t} + 4\left(\frac{3}{2} - \frac{3}{2}e^{-4t}\right) = 6,$$

and $y(0) = 3/2 - 3/2 = 0$, so our solution satisfies the initial-value problem.

25. First take Laplace transforms of both sides of the equation

$$\mathcal{L}\left[\frac{dy}{dt}\right] = 2\mathcal{L}[y] + 2\mathcal{L}[e^{-3t}]$$

and use the rules to simplify, obtaining

$$s\mathcal{L}[y] - y(0) = 2\mathcal{L}[y] + \frac{2}{s+3}$$

$$(s-2)\mathcal{L}[y] = y(0) + \frac{2}{s+3}$$

$$\mathcal{L}[y] = \frac{y(0)}{s-2} + \frac{2}{(s-2)(s+3)}.$$

Next note that

$$\mathcal{L}[y(0)e^{2t}] = y(0)/(s-2).$$

For the other summand, first simplify using partial fractions,

$$\frac{2}{(s-2)(s+3)} = \frac{A}{s-2} + \frac{B}{s+3}.$$

Putting the right-hand side over a common denominator gives $A(s+3) + B(s-2) = 2$, which can be written as $(A+B)s + (3A - 2B) = 2$. This yields $A + B = 0$ and $3A - 2B = 2$. Hence $B = -2/5$ and $A = 2/5$, and

$$\frac{2}{(s-2)(s+3)} = \frac{2/5}{s-2} - \frac{2/5}{s+3}.$$

Now, $\mathcal{L}[e^{2t}] = 1/(s-2)$ and $\mathcal{L}[e^{-3t}] = 1/(s+3)$ so

$$\mathcal{L}[y] = \frac{y(0)}{s-2} + \frac{2}{5}\frac{1}{s-2} - \frac{2}{5}\frac{1}{s+3}.$$

Hence,

$$y(t) = y(0)e^{2t} + \frac{2}{5}e^{2t} - \frac{2}{5}e^{-3t}.$$

The first two terms can be combined into one, giving

$$y(t) = ce^{2t} - \frac{2}{5}e^{-3t},$$

where $c = y(0) + 2/5$.