

Test 2: DIFFERENTIAL EQUATIONS

Math 341 Fall 2010
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Friday November 19
2:30pm-3:25pm

Name: _____

Directions: Read *all* problems first before answering any of them. There are 6 pages in this test. This is a 55-minute, limited-notes*, closed book, test. **No calculators.** You must show all relevant work to support your answers. Use complete English sentences as much as possible and **CLEARLY** indicate your final answers to be graded from your “scratch work.”

*You may use a one-sided 8.5” by 11” “cheat sheet” which must be stapled to the exam.

Offer: If there is a formula or piece of information that you feel that you need in order to solve a problem, I will provide it to you at a non-negotiable rate of at least a one point deduction.

Pledge: I, _____, pledge my honor as a human being and Occidental student, that I have followed all the rules above to the letter and in spirit.

No.	Score	Maximum
1		15
2		15
3		20
BONUS		4
Total		50

1. [15 points total.] **Linear Systems of Differential Equations, Trace-Determinant Plane, Bifurcation.** VISUAL & ANALYTIC.

Consider $\frac{d\vec{x}}{dt} = A\vec{x}$ where $A = \begin{bmatrix} \alpha & -\alpha/2 \\ 1 & -1 \end{bmatrix}$ and α is a known real-valued parameter. Recall

that the eigenvalues of the matrix A are given by $\lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2}$ where T is the trace of the matrix A and D is the determinant of matrix A .

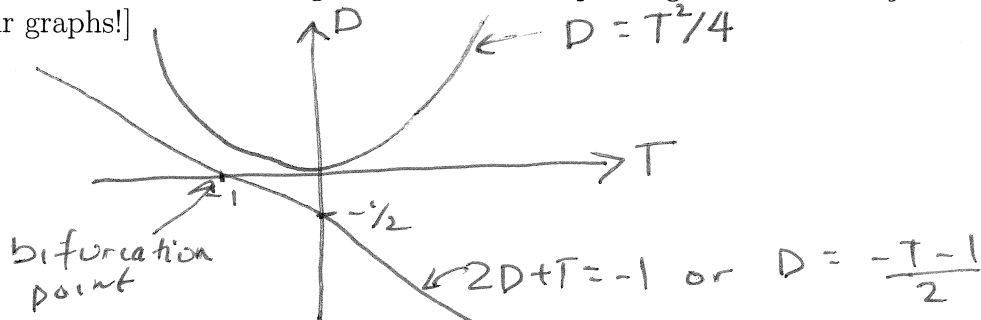
1(a) [5 points]. Compute the trace T and determinant D of matrix A for all values of α . Show that the relationship between the trace T and determinant D is $2D + T = -1$ regardless of the value of α .

$$T = \alpha - 1 \Rightarrow T + 1 = \alpha$$

$$D = -\alpha + \frac{\alpha}{2} = -\frac{\alpha}{2} \Rightarrow 2D = -\alpha = -(T+1) = -T - 1$$

$$\boxed{2D + T = -1}$$

1(b) [5 points]. Use your answer from (a) to sketch a graph in the trace-determinant plane depicting the relationship between the trace T and determinant D for the given matrix A as α changes. On the same axes, sketch the standard graph in the trace-determinant plane which separates the occurrence of real eigenvalues from complex eigenvalues for *any* matrix. [HINT: Label your graphs!]



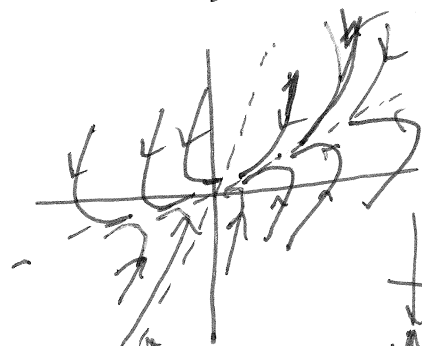
1(c) [5 points]. Does the qualitative nature of the phase portrait (and equilibrium at the origin) change as α varies? If so, **give all the bifurcation values of α , classify the equilibrium point at the origin** for values of α (greater than, less than and equal to the bifurcation value(s)) and **provide reasonable sketches of the phase portrait(s)** in each case in the space below.

$$D = 0 = -\frac{\alpha}{2} \Rightarrow \alpha = 0 = \alpha_B$$

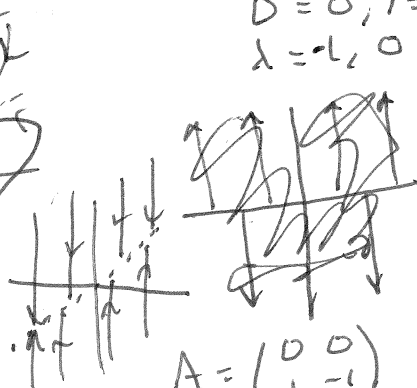
$$D < 0, \alpha > 0$$

$$\alpha < 0, D > 0, T < 0$$

$$\alpha = 0, D = 0, T = -1, \lambda = -1, 0$$

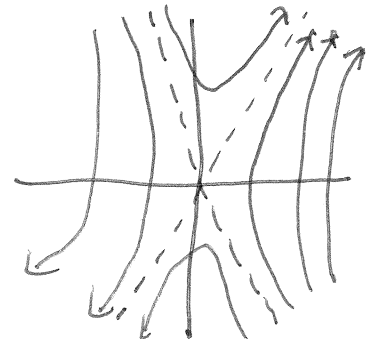


DOMINANT STABLE SOURCE



$$A = \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}$$

$$2 E_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} E_{-1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$



SADDLE (UNSTABLE)

2. [15 points total.] **Linearization, Hamiltonian function, Gradient function.** ANALYTIC, VERBAL & VISUAL.

Are the following statements TRUE or FALSE – put your answer in the box. To receive ANY credit, you must also give a brief, and correct, explanation in support of your answer! For example, if you think the answer is **FALSE** providing a counterexample for which the statement is NOT TRUE is best. If you think the answer is **TRUE** you should prove why you think the statement is always true. Your explanation of your answer is worth FOUR TIMES as much as the answer you put in the box. **For Full Credit you must write a full sentence explaining the reason for your choice of TRUE or FALSE.**

Consider the non-linear system

$$\begin{aligned} \frac{dx}{dt} &= \alpha x - y + y^3 = f(x,y) \\ \frac{dy}{dt} &= x + \alpha y + x^2 = g(x,y) \end{aligned}$$

where α is a known real-valued parameter.

2(a) **TRUE or FALSE?** “A Hamiltonian function $H(x, y)$ for the given nonlinear system of ODEs exists.”

$$\begin{aligned} f_x &= \alpha \\ g_y &= \alpha \end{aligned}$$

The Hamiltonian will only exist when $\alpha = 0$.

Hamiltonian Condition: $f_x = -g_y$

$$\alpha = -\alpha$$

true only if $\alpha = 0$
false when $\alpha \neq 0$

2(b) **TRUE or FALSE?** “A gradient function $G(x, y)$ for the nonlinear system of ODEs exists.”

$$\begin{aligned} f_y &= -1 + 3y^2 \\ g_x &= 1 + 2x \end{aligned}$$

Gradient condition: $f_y = g_x$

$$-1 + 3y^2 = 1 + 2x$$

Since the gradient condition does not hold for all (x, y) this system does NOT have a $G(x, y)$.

2(c) **TRUE or FALSE?** “The origin of the phase portrait for this nonlinear system will look like a center (a series of concentric circles) for all values of α .”

When $\alpha = 0$ is the only time the origin will look like a center.

$$J = \begin{pmatrix} \alpha & -1 + 3y^2 \\ 1 + 2x & \alpha \end{pmatrix}$$

$$J(0,0) = \begin{pmatrix} \alpha & -1 \\ 1 & \alpha \end{pmatrix} \quad \lambda^2 - \alpha\lambda + \alpha^2 - 1$$

$T = 2\alpha$
 $D = \alpha^2 + 1$
When $T = 0 = \alpha$
 λ will be complex and origin will be centers

3. [20 points total.] Linear Systems of Differential Equations, Matrix Exponential, General Solution. ANALYTIC & VERBAL.

Consider the initial value problem $\frac{d\vec{x}}{dt} = A\vec{x}$, $\vec{x}(0) = \vec{x}_0$ where $\vec{x} = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ and $\vec{x}_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$.

Usually we write the general solution as $\vec{x} = c_1 e^{\lambda_1 t} \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2$ where λ_i and \vec{v}_i are eigenvalues and eigenvectors of matrix A . However, the solution can also be written as $\vec{x}(t) = e^{At} \vec{x}_0$, which we will call the **matrix exponential solution**.

The goal of this problem is to show that the general solution to the given initial value problem (which we will call the **general eigenvector solution**) can be represented using the matrix exponential e^{At} .

Recall from Calculus that $\frac{d}{dt} e^{\square t} = \square e^{\square t}$ as long as $\frac{d}{dt} \square = 0$.

Recall from Linear Algebra that if A is diagonalizable, then the matrix exponential $e^{At} = S e^{\Lambda t} S^{-1}$ where S is an $n \times n$ matrix whose columns consist of the n eigenvectors $\vec{v}_1, \dots, \vec{v}_n$ of A , and Λ is an $n \times n$ diagonal matrix with the corresponding eigenvalues along the diagonal.

$$e^{At} = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \dots & \vec{v}_n \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & 0 & 0 & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 & 0 & 0 \\ 0 & 0 & e^{\lambda_3 t} & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & e^{\lambda_n t} \end{bmatrix} \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \dots & \vec{v}_n \end{bmatrix}^{-1}$$

NOTE: For this problem you can assume that A is a 2×2 diagonalizable matrix of the form $\begin{bmatrix} 0 & 1 \\ -q & -p \end{bmatrix}$ (where p and q are known fixed numbers) and A has two eigenvalues λ_1 and λ_2 with associated eigenvectors $\vec{v}_1 = \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}$.

3(a) [3 points]. Show that the **general eigenvector solution** satisfies the differential equation $\frac{d\vec{x}}{dt} = A\vec{x}$.

since $A\vec{v}_1 = \lambda_1 \vec{v}_1$
and $A\vec{v}_2 = \lambda_2 \vec{v}_2$

$$\vec{x} = c_1 e^{\lambda_1 t} \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2$$

$$\frac{d\vec{x}}{dt} = \lambda_1 c_1 e^{\lambda_1 t} \vec{v}_1 + \lambda_2 c_2 e^{\lambda_2 t} \vec{v}_2 = LHS$$

$$A\vec{x} = c_1 e^{\lambda_1 t} A\vec{v}_1 + c_2 e^{\lambda_2 t} A\vec{v}_2 = RHS$$

LHS = RHS

3(b) [3 points]. Show that in order for the **general eigenvector solution** to satisfy the given initial condition $\vec{x}(0) = \vec{x}_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$, the linear system $\begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ must be solved to find the unknown constants c_1 and c_2 . (HINT: What's a restriction involving λ_1 and λ_2 which must be satisfied?)

$$\vec{x}(0) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = c_1 \vec{v}_1 + c_2 \vec{v}_2 = c_1 \begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix}$$

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

This matrix is only invertible if
 $\det = \lambda_1 - \lambda_2 \neq 0$

3(c) [6 points]. Show that the **matrix exponential solution** $\vec{x}(t) = e^{At}\vec{x}_0$ satisfies the given initial value problem. [HINT: for a given solution to satisfy an initial value problem what must be true?]

$$\vec{x} = e^{At}\vec{x}_0$$

$$\frac{d\vec{x}}{dt} = A \underbrace{e^{At}\vec{x}_0}_{\vec{x}} = A\vec{x} \Rightarrow \text{The DE is satisfied}$$

$$t=0, \vec{x} = \vec{x}_0$$

$$\vec{x} = e^{A \cdot 0}\vec{x}_0$$

$$= I\vec{x}_0$$

$$= \vec{x}_0 \Rightarrow \text{The IC is satisfied}$$

3(d) [8 points]. Use your results from (a),(b) and (c) to show that the **matrix exponential solution** is identical to the **general eigenvector solution** for the given initial value problem

$\frac{d\vec{x}}{dt} = A\vec{x}$, $\vec{x}(0) = \vec{x}_0$ where A has two eigenvalues λ_1 and λ_2 with associated eigenvectors $\vec{v}_1 =$

$\begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}$. (HINT: What's a restriction involving λ_1 and λ_2 which must be satisfied?)

EXPLAIN YOUR ANSWER(S) and SHOW ALL YOUR WORK.

$$e^{At}\vec{x}_0 \stackrel{?}{=} c_1 e^{\lambda_1 t} \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2$$

$$\begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix}^{-1} \begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix} \stackrel{?}{=} c_1 e^{\lambda_1 t} \begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix} + c_2 e^{\lambda_2 t} \begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \stackrel{?}{=} \begin{pmatrix} c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \\ c_1 \lambda_1 e^{\lambda_1 t} + c_2 \lambda_2 e^{\lambda_2 t} \end{pmatrix}$$

$$\begin{bmatrix} e^{\lambda_1 t} & e^{\lambda_2 t} \\ \lambda_1 e^{\lambda_1 t} & \lambda_2 e^{\lambda_2 t} \end{bmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \stackrel{?}{=} \begin{pmatrix} c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \\ c_1 \lambda_1 e^{\lambda_1 t} + c_2 \lambda_2 e^{\lambda_2 t} \end{pmatrix}$$

$$\begin{pmatrix} c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \\ c_1 \lambda_1 e^{\lambda_1 t} + c_2 \lambda_2 e^{\lambda_2 t} \end{pmatrix} \stackrel{?}{=} \begin{pmatrix} c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \\ c_1 \lambda_1 e^{\lambda_1 t} + c_2 \lambda_2 e^{\lambda_2 t} \end{pmatrix}$$

$$\text{LHS} = \text{RHS} \checkmark$$

Note $\begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix}$ is only invertible iff $\lambda_1 \neq \lambda_2$

BONUS. [4 points]. Find the (matrix exponential) solution to the initial value problem

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \vec{x}, \quad \vec{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and simplify it into the form } \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}. \quad \begin{matrix} T = -3 \\ \Delta = 2 \end{matrix}$$

CHECK YOUR ANSWER!

HINT: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$$\lambda^2 + 3\lambda + 2 = 0$$

$$(\lambda + 2)(\lambda + 1) = 0$$

$$\lambda = -1, -2$$

$$\vec{x}(t) = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} e^{-t} & e^{-2t} \\ -e^{-t} & -2e^{-2t} \end{pmatrix} \frac{1}{-1} \begin{pmatrix} -2 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} e^{-t} & e^{-2t} \\ -e^{-t} & -2e^{-2t} \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} e^{-t} & e^{-2t} \\ -e^{-t} & -2e^{-2t} \end{pmatrix} \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

$$\vec{x}(t) = \begin{pmatrix} 3e^{-t} - 2e^{-2t} \\ -3e^{-t} + 4e^{-2t} \end{pmatrix}$$

$$t=0, \quad \vec{x}(0) = \begin{pmatrix} 3-2 \\ -3+4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \checkmark \quad (1C)$$

$$\vec{x}' = \begin{pmatrix} -3e^{-t} + 4e^{-2t} \\ 3e^{-t} - 8e^{-2t} \end{pmatrix} \stackrel{?}{=} \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} \begin{pmatrix} 3e^{-t} - 2e^{-2t} \\ -3e^{-t} + 4e^{-2t} \end{pmatrix}$$

$$= \begin{pmatrix} -3e^{-t} + 4e^{-2t} \\ -6e^{-t} + 4e^{-2t} + 9e^{-t} - 12e^{-2t} \end{pmatrix}$$

$$= \begin{pmatrix} -3e^{-t} + 4e^{-2t} \\ 3e^{-t} - 8e^{-2t} \end{pmatrix}$$

$$\text{LHS} = \text{RAS} \checkmark \quad (DE)$$