Test 1: DIFFERENTIAL EQUATIONS

Math 341 Fall 2014 — Wednesday October 8

 — 3:00-3:55pm

Name: BUCKMIRE

Instructions:

Read *all* problems first before answering any of them. This tests consists of three (3) problems (and a BONUS problem) on six (6) pages.

The topic of the problem is **in bold**, the number of points each problem is worth is in *italics* and the kind of skills required to solve each problem are in ALL CAPS.

This is a 55-minute, closed notes, closed book, test. **No calculators or electronic devices may be used.**

You must show all relevant work to support your answers. Use complete English sentences as much as possible and CLEARLY indicate your final answers to be graded from your "scratch work."

Questions: FEEL FREE TO ASK CLARIFICATION QUESTIONS AT ANY TIME!

Pledge: I, ______________________________, pledge my honor as a human being and Occidental student, that I have followed all the rules above to the letter and in spirit.

No.	Score	Maximum
1		40
2		30
3		30
BONUS		10
Total		**100**

1. *[40 points total.]* **Existence and Uniqueness Theorem, Functions, Equilibrium Solutions, Separation of Variables, Interval of Validity.** ANALYTIC & VERBAL.

Determine whether the following statements are **TRUE** or **FALSE** and place your answer in the box. To receive FULL credit, you must also give a brief, and correct, explanation in support of your answer! The explanation for your answer is worth EIGHT POINTS while your TRUE or FALSE answer is worth 2 points. Try to present as much information and understanding of the question and your answer in your short written response.
For all parts of this question, consider the initial value problem

$$\frac{dy}{dx} = xy^{1/3}, \quad y(0) = A, \text{ where } A \text{ is any real number.}$$

1(a) *[10 points].* **TRUE or FALSE**: "The initial value problem $y' = xy^{1/3}$, $y(0) = A$ has the exact solution $y(x) = \left(\frac{x^2}{3} + A^{2/3}\right)^{3/2}$ for all values of A."

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$\boxed{\text{TRUE}}$

Check: $x = 0, y = A$
IC $x = 0,\ y(0) = (A^{2/3})^{3/2} = A$ ✓

Check DE $\frac{dy}{dx} = \frac{3}{2}\left(\frac{x^2}{3} + A^{2/3}\right)^{1/2} \cdot \frac{2x}{3}$

$= \left(\frac{x^2}{3} + A^{2/3}\right)^{1/2} \cdot x$

$= y^{1/3} \cdot x$ ✓

$\frac{dy}{dx} = xy^{1/3}$

$\frac{dy}{y^{1/3}} = x\,dx$

$\int y^{-1/3}dy = \int x\,dx$

$\frac{3y^{2/3}}{2} = \frac{x^2}{2} + C$

$y^{2/3} = \frac{x^2}{3} + D$ when $x = 0, y = A$
$A^{2/3} = D \Rightarrow y^{2/3} = \frac{x^2}{3} + A^{2/3} \Rightarrow y = (x^2/3 + A^{2/3})^{3/2}$

1(b) *[10 points].* **TRUE or FALSE**: "The initial value problem $y' = xy^{1/3}$, $y(0) = A$ has an equilibrium solution $y(x) = 0$ when $A \neq 0$."

$\boxed{\text{FALSE}}$

If $A \neq 0$ then $y(x) = 0$ is not a solution of the given IVP
$y' = xy^{1/3}$, $y(0) = A$

If $A = 0$, then $y(x) = 0$ IS an equilibrium solution to the IVP.
So is $y = \frac{x^3}{\sqrt{27}}$.

This IVP does not have unique solutions when $A = 0$.

1(c) *[10 points]*. **TRUE or FALSE**: "The initial value problem $y' = xy^{1/3}$, $y(0) = A$ has a unique solution when $A \neq 0$."

$\boxed{\text{TRUE}}$

Existence & Uniqueness Theorem

$f(x,y) = xy^{1/3}$ at $x=0, y=A$ $(A \neq 0)$

This function is continuous for all $(x,y) \in \mathbb{R}$ since it is product of two continuous functions.

$\frac{\partial f}{\partial y} = \frac{1}{3}xy^{-2/3}$ This function is NOT continuous along $y=0$ but since $A \neq 0$ that is not in our ~~set~~ of possibilities.

So, by EUT $y' = xy^{1/3}, y(0) = A \neq 0$ has a unique sol^n (given in part (b))

1(d) *[10 points]*. **TRUE or FALSE**: "The initial value problem $y' = xy^{1/3}$, $y(0) = A$ has the same interval of validity regardless of the value of A."

$\boxed{\text{TRUE}}$

The interval of validity has nothing to do with the output of the initial condition.

The solution is $y(x) = \left(\frac{x^2}{3} + A^{2/3}\right)^{3/2}$ and has a domain which is all x values, which is the same set as the ~~domain~~ of interval of validity.

2. *[30 points total.]* **Slope Fields, Solution Techniques for Linear ODEs, Autonomous DEs, Non-homogeneous DEs.** ANALYTIC, VERBAL & VISUAL.

2(a) *[10 points].* Find the general solution of **(ODE A)** $\frac{dy}{dt} = t^2 + y$.

STATE YOUR SOLUTION TECHNIQUE (WHICH MUST BE DIFFERENT FROM PART(B)).

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$y_p(t) = At^2 + Bt + C$

$y_p' = 2At + B = LHS$

$t^2 + y_p = t^2 + At^2 + Bt + C = RHS$

$\Rightarrow \theta(t^2)$ $\quad 0 = A + 1 \quad A = -1$

$\theta(t)$ $\quad 2A = B \quad B = -2$

$\theta(1)$ $\quad B = C \quad C = -2$

$y_p(t) = -t^2 - 2t - 2$

$y_h(t) = Ke^t$

$\boxed{y = Ke^t - t^2 - 2t - 2}$

INTEGRATING FACTOR

$y' - y = t^2$

$\mu = e^{\int -1\,dt} = e^{-t}$

$(ye^{-t})' = t^2 e^{-t}$

$ye^{-t} = \int t^2 e^{-t}\,dt$

$= \frac{e^{-t}}{-1}(t^2 + 2t + 2) + C$

$y = -1(t^2 + 2t + 2) + Ce^t$

$\boxed{y = -t^2 - 2t - 2 + Ce^t}$

Possibly useful formula: $\int t^2 e^{at}\,dt = \frac{1}{a^3} e^{at}(a^2t^2 - 2at + 2)$

2(b) *[10 points].* Find the general solution of **(ODE B)** $\frac{dy}{dt} = t^2 + 1$.

STATE YOUR SOLUTION TECHNIQUE (WHICH MUST BE DIFFERENT FROM PART (A))

Separation of variables

$\frac{dy}{dt} = t^2 + 1$

$y(t) = \frac{t^3}{3} + t + C$

Check ODE B

$y' = \frac{3t^2}{3} + 1$

$= t^2 + 1$ ✓

check ODE A

$y = Ke^t - t^2 - 2t - 2$

$y' = Ke^t - 2t - 2$

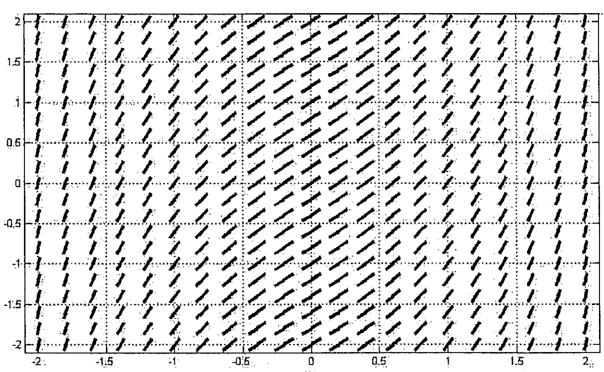
$RHS = y + t^2 = Ke^t + t^2 - t^2 - 2t - 2$

$= Ke^t - 2t - 2$

$y + t^2 = RHS = y'$ ✓

2(c) *[10 points].* Select whether the slope field below corresponds to **(ODE A)** or **(ODE B)** and EXPLAIN THE REASON FOR YOUR CHOICE.

$\boxed{\text{ODE B}}$



Since the slope field only depends on t (it varies in the horizontal direction is constant in vertical) $y' = t^2 + 1$ is the correct choice

3. *[30 pts. total]* **Phase Lines, Equilibria, Bifurcations, Geometric Representations.** VISUAL & ANALYTIC.

Consider the following differential equation: $\dfrac{dy}{dt} = \sqrt{y} + \alpha$, where α is a real-valued parameter and $\sqrt{y}$ outputs only the positive square root of y.

3(a) *[10 points]* What are the equilibrium values of the differential equation? Identify them as y^*. They should depend on values of the parameter α.

$\sqrt{y} + \alpha = 0 \Rightarrow \sqrt{y} = -\alpha$ But $\sqrt{y}$ is always ≥ 0 so no solutions when $\alpha > 0$

when $\alpha < 0$, $\sqrt{y^*} = -\alpha$

or $y^* = \alpha^2$

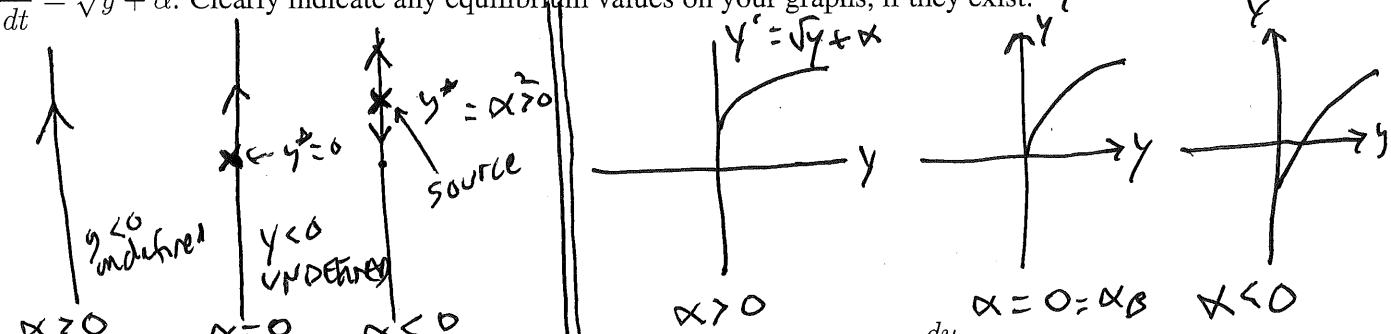
ONLY DO ONE OF THE FOLLOWING TWO QUESTIONS LABELLED 3(b)

3(b) *[10 points]* Is there a bifurcation value for the parameter α? If so, call it α_B and draw phase lines corresponding to the cases where $\alpha < \alpha_B$, $\alpha = \alpha_B$ and $\alpha > \alpha_B$. Indicate locations (and values) of y^* on your phase lines.

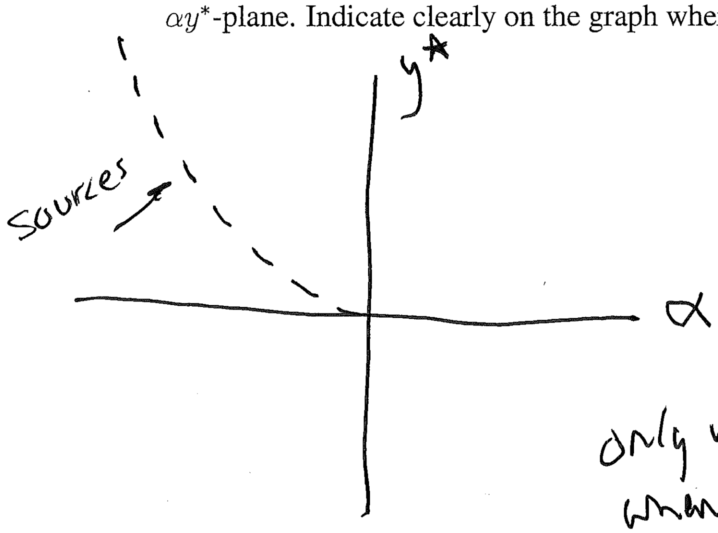
OR

3(b) *[10 points]* Is there a bifurcation value for the parameter α? If so, call it α_B and sketch graphs of $f(y;\alpha)$ versus y corresponding to the cases where $\alpha < \alpha_B$, $\alpha = \alpha_B$ and $\alpha > \alpha_B$ where $\dfrac{dy}{dt} = \sqrt{y} + \alpha$. Clearly indicate any equilibrium values on your graphs, if they exist.

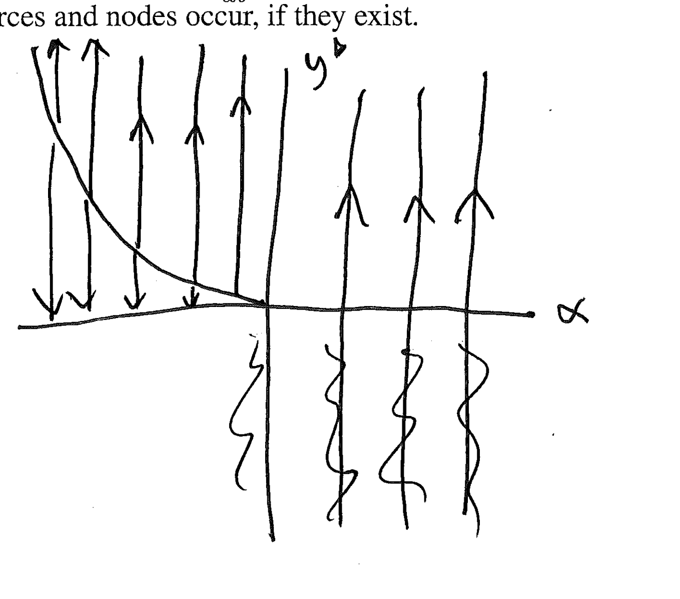
$B = 0$ because no equilibria then a source appears



3(c) *[10 points]* Draw a bifurcation diagram for the differential equation $\dfrac{dy}{dt} = \sqrt{y} + \alpha$, in the αy^*-plane. Indicate clearly on the graph where sinks, sources and nodes occur, if they exist.



OR



only valid when $y^* \geq 0$

BONUS. *[10 points]* **Euler's Method for Systems.** ANALYTIC & COMPUTATIONAL.
Find an approximate value of the solution $y(t)$ to the following initial value problem

$$y'' + py' + qy = 0 \qquad \text{where } y(3) = A, \quad y'(3) = B$$

after two time steps of size Δt, in other words obtain an expression involving the real-valued parameters Δt, A, B, p and q which is an estimate of $y(3 + 2\Delta t)$.

let $\vec{x} = \begin{pmatrix} y \\ y' \end{pmatrix}$ $\quad \frac{d\vec{x}}{dt} = \begin{pmatrix} \frac{dy}{dt} \\ \frac{d^2y}{dt^2} \end{pmatrix} = \begin{pmatrix} y' \\ -py' - qy \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix}\begin{pmatrix} y \\ y' \end{pmatrix}$

$$\frac{d\vec{x}}{dt} = M\vec{x}, \quad \vec{x}(3) = \begin{pmatrix} A \\ B \end{pmatrix}$$

Euler's Method for Systems when $\vec{F}(\vec{x}) = M\vec{x}$

$$\vec{x}_{NEW} = \vec{x}_{OLD} + (M\vec{x}_{old}) \cdot \Delta t$$
$$= (I + M\Delta t)\vec{x}_{OLD}$$

$$\vec{x}_1 = (I + M\Delta t)\begin{pmatrix} A \\ B \end{pmatrix} = (I + M\Delta t)\vec{x}_0$$

$y'' = -py' - qy$

$$\vec{x}_2 = (I + M\Delta t)\vec{x}_1 = (I + M\Delta t)^2\begin{pmatrix} A \\ B \end{pmatrix}$$
$$= \begin{pmatrix} 1 & \Delta t \\ -q\Delta t & 1 - p\Delta t \end{pmatrix}^2 \begin{pmatrix} A \\ B \end{pmatrix}$$

t	y	y'	$\Delta y = y'\Delta t$	$\Delta y' = y'' \cdot \Delta t = -py'\Delta t - qy\Delta t$
3	A	B	$B\Delta t$	$(-pB - qA)\Delta t$
$3+\Delta t$	$A + B\Delta t$	$B - \Delta t(pB + qA)$	$B\Delta t - (\Delta t)^2(pB + qA)$	$-p\Delta t(B - \Delta t(pB+qA)) - q\Delta t(A + B\Delta t)$
$3+2\Delta t$	$A + 2B\Delta t - (\Delta t)^2(pB+qA)$			

$$\boxed{y(3 + 2\Delta t) \approx A + 2B\Delta t - (\Delta t)^2(pB + qA)}$$

$$\begin{pmatrix} 1 & \Delta t \\ -q\Delta t & 1 - p\Delta t \end{pmatrix}\begin{pmatrix} 1 & \Delta t \\ -q\Delta t & 1 - p\Delta t \end{pmatrix}$$
$$= \begin{pmatrix} 1 - q\Delta t^2 & 2\Delta t - p\Delta t^2 \\ -2q\Delta t + pq\Delta t^2 & -q\Delta t^2 + (1 - p\Delta t)^2 \end{pmatrix}$$

$$(I + M\Delta t)^2\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} A - qA(\Delta t)^2 + 2B\Delta t - pB\Delta t^2 \\ y'(3 + 2\Delta t) \end{pmatrix} \qquad (I + M\Delta t)^2$$