# **FINAL EXAM:** Differential Equations

Math 341 Fall 2014 — Thursday December 11
 — 6:30pm-9:30pm

# Name: ______________________________

**Directions**: Read *all* **ten problems** first before answering any of them. There are **13** pages in this test. This exam is designed to be a 2-hour cumulative exam on the central ideas, techniques and methods of the course. You have three hours to complete the entire exam. **No calculators.** You must show all relevant work to support your answers. You must work alone. Use complete English sentences as much as possible and CLEARLY indicate your final answers to be graded from your "scratch work."

You may consult a 8.5" by 11" "cheat sheet" with writing on both sides. There is a formula sheet at the end of The Exam which includes all the formulas you should need for Laplace Transforms.

**Pledge:** I, ______________________________, pledge my honor as a human being and Occidental student, that I have followed all the rules above to the letter and in spirit.

| No. | Score | Maximum |
|---|---|---|
| 1 | | 10 |
| 2 | | 10 |
| 3 | | 10 |
| 4 | | 10 |
| 5 | | 10 |
| 6 | | 10 |
| 7 | | 10 |
| 8 | | 10 |
| 9 | | 10 |
| 10 | | 10 |
| BONUS | | 10 |
| **Total** | | **100** |

**1** *[10 points total.]* **Existence and Uniqueness Theorem.**

**(a)** Suppose $y' = f(x,y)$ where $f$ and $\dfrac{\partial f}{\partial y}$ are continuous everywhere, i.e. at all $(x,y)$ values. Does this mean that any solution $y(x)$ to the ODE must be continuous everywhere? EXPLAIN YOUR ANSWER.

**(b)** Suppose $y' = xy^{1/3}$. What does the Existence and Uniqueness Theorem allow us to conclude about solutions to the ODE for initial conditions where $y = 0$ for some value of $x$? EXPLAIN YOUR ANSWER.

**2** *[10 points total.]* **Bifurcation, Phase Lines, Equilibria, Stability.**

Consider the autonomous ordinary differential equation $y' = ay^2 + y^4$ where $a$ is an arbitrary real-valued parameter. Find the equilibria values $y^*$ and compute the bifurcation value $a_B$. Sketch phase lines for representative values $a < a_B$, $a = a_B$ and $a > a_B$. Clearly label any sinks, sources or nodes. Finally, sketch the bifurcation diagram in the $ay^*$-plane.

**3** *[10 points total.]* **Hamiltonian and Gradient Systems.**

Consider $\phi(x, y) = \dfrac{x^2}{2} - \dfrac{x^4}{4} - \dfrac{y^2}{2} + 8.$

**(a)** Write down a system of differential equations for which the given $\phi(x, y)$ is a Gradient function. CONFIRM your given system is indeed a Gradient system.

**(b)** Write down a system of differential equations for which the given $\phi(x, y)$ is a Hamiltonian function. CONFIRM your given system is indeed a Hamiltonian system.

**4** *[10 points total.]* **Equilibria of Nonlinear Systems, Linearization, Jacobian, Eigenvalues.** Consider the following nonlinear system of differential equations

$$\begin{aligned} \frac{dx}{dt} &= -2x + 2x^2 \\ \frac{dy}{dt} &= -3x + y + 3x^2 \end{aligned}$$

Find and classify all the equlibrium points of the system.

**5** *[10 points total.]* **Solving Linear System of ODEs.**
Solve $\dfrac{d\vec{x}}{dt} = \begin{bmatrix} -2 & 3 \\ 3 & -2 \end{bmatrix} \vec{x}.$

**6** *[10 points total.]* **Nonhomogeneous and Homogeneous Problems.**

**(a)** Solve $t\dfrac{dy}{dt} + 2y = 0$. SHOW ALL YOUR WORK!

**(b)** Solve $t\dfrac{dy}{dt} + 2y = 2t^2, \quad y(2) = 1$. SHOW ALL YOUR WORK!

**9** *[10 points total.]* **Laplace Transforms and Initial Value Problems.**
Consider the following initial value problem, whose exact solution $y(t)$ is the zeroth-order Bessel Function of the First Kind, denoted $J_0(t)$

$$ty'' + y' + ty = 0, \quad y(0) = 1, \quad y'(0) = 0.$$

Given that $\mathcal{L}[ty'] = -\frac{d}{ds}\{\mathcal{L}[y']\} = -\frac{d}{ds}\{sY(s) - y(0)\} = -\{\frac{ds}{ds}Y(s) + s\frac{dY}{ds} - \frac{d}{ds}y(0)\} = -s\frac{dY}{ds} - Y$, show that applying the Laplace Transform to the given initial value problem produces the following equation for $Y(s)$.

$$\mathcal{L}[ty'' + y' + ty] = -(s^2 + 1)\frac{dY}{ds} - sY = 0$$

[HINT: Since you are given $\mathcal{L}[ty']$, how can you use that information to find expressions for $\mathcal{L}[ty'']$ and $\mathcal{L}[ty]$? Think about what multiplication in $t$-space means when Laplace-transformed into $s$-space.]

**10** *[10 points total.]* **Separation of Variables.**
Consider the differential equation from **9**,

$$-(s^2+1)\frac{dY}{ds} - sY = 0$$

Let $Y(0) = A$ where $A$ is a (un)known real value. Solve this initial value problem for $Y(s)$. (SEE BONUS PROBLEM to determine $A$!) **Check that your solution satisfies the IVP.**

**BONUS QUESTION** *[10 points total.]*
**(a)** Given that $y(t)$ and its Laplace Transform $Y(s)$ satisfy the relationship

$$\lim_{s \to \infty} sY(s) = y(0)$$

use this result to obtain the value of the unknown constant $A$ from **10**.

**(b)** From **9**, **10** and **part (a)** above, what is an explicit functional form for the Laplace Transform of the Zeroth-Order Bessel's Function of the First Kind, i.e. $\mathcal{L}[J_0(t)]$? Recall, $J_0(t)$ satisfies the initial value problem from **9**: $ty'' + y' + ty = 0, \quad y(0) = 1, \quad y'(0) = 0.$