Complex Analysis

Math 312 MWF 10:30am - 11:25am

 Fowler 112

Class 33 **(Monday April 20)**

SUMMARY Cauchy Principal Value of an Improper Integral
CURRENT READING Brown & Curchill pages 204–208
NEXT READING Work on your term projects!

Exercise

Consider the improper integral $\int_{-\infty}^{\infty} \frac{-2}{x^3}\,dx$
What do you have to do before you can evaluate the integral?

Is this the same value as $\lim_{R\to\infty} \int_{-R}^{R} \frac{-2}{x^3}\,dx$?

Cauchy Principal Value

The Cauchy Principal Value of an improper integral, denoted by P.V. $\int_{-\infty}^{\infty} f(x)\,dx$ is defined as

$$\text{P.V.} \int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx$$

Groupwork

Find p.v.$\int_{-\infty}^{\infty} x\,dx$ and p.v.$\int_{-\infty}^{\infty} x^2\,dx$

Compare these answers to the improper integral $\int_{-\infty}^{\infty} x\,dx$ and $\int_{-\infty}^{\infty} x^2\,dx$

What's the difference between two sets of answers? Notice any patterns?

The relationship between the Cauchy Principal value of an improper integral and the improper integral can be sumarized as

convergence of $\int_{-\infty}^{\infty} f(x)\, dx$ IMPLIES p.v. $\int_{-\infty}^{\infty} f(x)\, dx$ EXISTS

p.v. $\int_{-\infty}^{\infty} f(x)\, dx$ EXISTS DOES NOT IMPLY convergence of $\int_{-\infty}^{\infty} f(x)\, dx$

There is a condition on $f(x)$ from which we will know when the two values are equal: If $f(x)$ is an EVEN FUNCTION or if the improper integral converges.

p.v. $\int_{-\infty}^{\infty} f(x)\, dx$ IS EQUAL TO $\int_{-\infty}^{\infty} f(x)\, dx,$ when $f(x) = f(-x)$

We have already learned how to evaluate improper integrals using contour integrals. Now we can be more precise and realize we were actually evaluating the principal value of the improper integral. (All the examples I gave you were of even functions.)

Evaluating Improper Integrals using Contour Integration

$$\text{p.v.} \int_{-\infty}^{\infty} f(x)\, dx = \int_{-R}^{R} f(x) dx = \oint f(z)\, dz = 2\pi i \sum Res(f)$$

As long as $f(z)$ obeys the two boundedness theorems such that $|f(z)| < M/|z|^k$ where $k > 1$. (In other words if $f(z)$ is a rational function $p(z)/q(z)$ then the degree of $q(z)$ must be greater than degree of $p(z) + 1$.)

Exercise

Find the Cauchy principal value of $\int_{-\infty}^{\infty} \frac{dx}{x^2 + 2x + 2}$ by evaluating a related contour integral for a function $f(z)$.

(Why do you know you can use this method?)

(Is the value of the integral the same as the cauchy principal value? Why/why not?)

NOTE: The Final Exam is on Wednesday April 29th at 1pm