Complex Analysis

Math 312 MWF 10:30am - 11:25am

 Fowler 112

Class 31 **(Friday April 10)**

SUMMARY Applications of Residues to Real Integrals
CURRENT READING Brown & Curchill pages 210–219
NEXT READING Brown & Curchill 204–210

The beauty of Complex Residue Calculus is that it allows us to evaluate a vast number of contour integrals. In fact, we can show that we can use residues to evaluate associated **real** integrals which would otherwise be very difficult to get exact values for become quite easy as contour integrals.

Recall the definition of $z = e^{i\theta} = \cos(\theta) + i\sin(\theta)$ Therefore, we can write $\cos(\theta)$ and $\sin(\theta)$ in terms of z.
$\cos\theta =$ $\sin\theta =$

Exercise

Rewrite the integral $\displaystyle\int_0^{2\pi} \frac{d\theta}{3 + 2\sin\theta}$ in terms of z, using $z = e^{i\theta}$ where $0 \le \theta \le 2\pi$.

Evaluate the real integral by evaluating the contour integral.

GROUPWORK

Show that $\displaystyle\int_0^{2\pi} \frac{\cos 3\theta}{5 - 4\cos\theta} d\theta = \frac{i}{2}\oint_{|z|=1} \frac{z^6 + 1}{z^3(2z^2 - 5z + 2)} dz$

You should be able to compute the value of the contour integral and show it equals $\dfrac{\pi}{12}$.

Evaluation of real integrals of the form $\int_{-\infty}^{\infty} f(x)\,dx$

We can also evaluate **Improper Integrals** more easily by evaluating associated contour integrals. however, we have to have some conditions on the integrand $f(z)$.

Boundedness Theorem 1

If $|f(z)| \leq \dfrac{M}{R^k}$ for $z = Re^{i\theta}$ where $k > 1$ and M are constants and C_R is the contour consisting of the semi-circle of radius R from $\theta = 0$ to $\theta = \pi$, then $\lim\limits_{R\to\infty} \int_{C_R} f(z)e^{inz}\,dz = 0$

Boundedness Theorem 2

If $|f(z)| \leq \dfrac{M}{R^k}$ for $z = Re^{i\theta}$ where $k > 1$, $n > 0$ and M are constants and C_R is the contour consisting of the semi-circle of radius R from $\theta = 0$ to $\theta = \pi$, then $\lim\limits_{R\to\infty} \int_{C_R} f(z)e^{inz}\,dz = 0$

The second boundedness theorem is sometimes called **Jordan's Lemma**.

You can use these theorems above to say that

$$\int_{-R}^{R} f(x)dx + \int_{C_R} f(z)\,dz = \oint_C f(z)\,dz$$

and then take the limit as $R \to \infty$ to say that

$$\int_{-\infty}^{\infty} f(x)\,dx = \oint f(z)\,dz = 2\pi i \sum Res(f)$$

if the boundedness theorems above apply to $f(z)$ (since then $\int_{C_R} f(z)\,dz \to 0$ as $R \to \infty$)

Examples

Show that $\displaystyle\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^2}\,dx = \frac{\pi}{2}$

Show that (for $a > 0$) $\displaystyle\int_0^{\infty} \frac{\cos(ax)}{x^2+1}\,dx = \frac{\pi}{2}e^{-a}$