Complex Analysis

Math 312 MWF 10:30am - 11:25am

 Fowler 112

Class 29 **(Monday April 6)**

SUMMARY Classifying Singularities and Representing Functions using Series
CURRENT READING Brown & Curchill pages 150-154
NEXT READING Brown & Curchill pages 138-156

Cauchy's Second Residue Theorem

If a function $f(z)$ is analytic everywhere in the finite plane except for a finite number of singular points interior to a positively oriented simple closed contour C, then

$$\oint f(z)\, dz = 2\pi i \mathbf{Res}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right) \mathbf{;\, 0}\right]$$

In other words, instead of finding the residues of all the singularities of the given function $f(z)$ which lie inside the given contour C, all you need to do is find the residue at a single point, $z = 0$, of the associated function $\frac{1}{z^2} f(\frac{1}{z})$. Note what's really going on involves finding the residue of the function at the point at infinity.

Exercise

Evaluate $\oint_{|z|=2\pi} \tan(z)\, dz$ using Cauchy's Second Residue Theorem.

GROUPWORK

Evaluate $\oint_{|z|=2} \frac{3z+2}{z^2+1}\, dz$ using Cauchy's Second Residue Theorem.

Classifying Singularities

There are basically three types of singularities (points where $f(z)$ is not analytic) in the complex plane. They are called **removable singularities**, **isolated singularities** and **branch singularities**.

Isolated Singularity

An isolated singularity of a function $f(z)$ is a point z_0 such that $f(z)$ is analytic on the punctured disc $0 < |z - z_0| < r$ but is *undefined* at $z = z_0$. We usually call isolated singularities **poles**. An example is $z = i$ for the function $z/(z - i)$.

Removable Singularity

A removable singularity is a point z_0 where the function $f(z_0)$ appears to be undefined but if we assign $f(z_0)$ the value w_0 with the knowledge that $\lim_{z \to z_0} f(z) = w_0$ then we can say that we have "removed" the singularity. An example would be the point $z = 0$ for $f(z) = \sin(z)/z$.

Branch Singularity

A branch singularity is a point z_0 through which all possible branch cuts of a multi-valued function can be drawn to produce a single-valued function. An example of such a point would be the point $z = 0$ for $\operatorname{Log}(z)$.

There is also a special kind of isolated singularity, called an **essential singularity**. The canonical example of an essential singularity is $z = 0$ for the function $f(z) = e^{1/z}$. The easiest way to define an essential singularity of a function involves Laurent Series (see below).

Laurent series

In fact, the best way to identify an essential singularity z_0 of a function $f(z)$ (and an alternative way to compute residues) is to look at the **series representation** of the function $f(z)$ about the point z_0

That is,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}, \qquad R_1 < |z - z_0| < R_2$$

This formula for a Laurent series is sometimes written as

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n \qquad \text{where } c_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz, \quad n = \pm 1, \pm 2, \ldots$$

This first part of this series should look somewhat familiar from your experience with real functions, since the expression is clearly a **Taylor series** if $b_n = 0$ for all n. This first part of the series representation is known as the **analytic part** of the function. The second part (with the *negative exponents*) is called the **principal part** of the function. However if a_n and b_n are not all identically zero this type of series is called a **Laurent series** and converges to the function $f(z)$ in the annular region $R_1 < |z - z_0| < R_2$.

Exercise

Let's show why expressing the function $f(z)$ in terms of a Laurent Series is useful by proving that the value of the **Res**$(f; z_0)$ is exactly equal to b_{-1} (or c_{-1}), that is, the coefficient of the $\dfrac{1}{z - z_0}$ term. We can do this by integrating the Laurent series term by term on some closed contour C and using the CIF.

Review of Sequences and Series

Recall that an infinite **sequence** $\{z_n\}$ converges to z if for each $\epsilon > 0$ there exists an N such that if $n > N$ then $|z_n - z| < \epsilon$

The sequence $z_1, z_2, z_3, \ldots, z_n, \ldots$ converges to the value $z = x + iy$ if and only if the sequence $x_1, x_2, x_3, \ldots$ converges to x and $y_1, y_2, y_3, \ldots$ converges to y.

In other words $\lim_{n\to\infty} z_n = z \Leftrightarrow \lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$

An infinite **series** $\sum_{n=1}^{\infty} z_n = z_1 + z_2 + z_3 + \cdots + z_n + \cdots$ converges to S if the sequence S_N of **partial sums** where $S_N = z_1 + z_2 + z_3 + z_4 + \cdots + z_N \qquad (N = 1, 2, 3, \ldots)$ converges to S. Then we say that $\sum_{n=1}^{\infty} z_n = S$.

As with sequences, series can be split up into real and imaginary parts. Suppose $z_n = x_n + iy_n$ and $\sum_{n=1}^{\infty} z_n = Z$, $\sum_{n=1}^{\infty} x_n = X$ and $Y = \sum_{n=1}^{\infty} y_n$ then $Z = X + iY$.

Taylor series Suppose a function f is analytic throughout an open disk $|z - z_0| < R_0$ centered at z_0 with radius R_0. Then at each point z in this disk $f(z)$ has the series representation

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \qquad \text{where } a_n = \frac{f^{(n)}(z_0)}{n!} \text{ for } (n = 0, 1, 2, \ldots)$$

In other words the function $f(z)$ can be represented exactly by the infinite series in the disk $|z - z_0| < R$

When $z_0 = 0$ the series is known as a **Maclaurin series**.

Here are some examples of well known Maclaurin series you should know.

$$
\begin{array}{lclcll}
e^z & = & 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots & = & \sum_{k=0}^{\infty} \frac{z^k}{k!} & |z| < \infty \\
\sin(z) & = & z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots & = & \sum_{k=0}^{\infty} \frac{(-1)^{k+1} z^{2k+1}}{(2k+1)!} & |z| < \infty \\
\cos(z) & = & 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \ldots & = & \sum_{k=0}^{\infty} \frac{(-1)^{k+1} z^{2k}}{(2k)!} & |z| < \infty \\
\sinh(z) & = & z + \frac{z^3}{3!} + \frac{z^5}{5!} + \ldots & = & \sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)!} & |z| < \infty \\
\cosh(z) & = & 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \ldots & = & \sum_{k=0}^{\infty} \frac{z^{2k}}{(2k)!} & |z| < \infty \\
\frac{1}{1-z} & = & 1 + z + z^2 + z^3 + \ldots & = & \sum_{k=0}^{\infty} z^k & |z| < 1 \\
\ln(1+z) & = & z - \frac{z^2}{2} + \frac{z^3}{3} + \ldots & = & \sum_{k=1}^{\infty} \frac{z^k}{k} & |z| < 1 \\
(1+z)^p & = & 1 + pz + \frac{p(p-1)z^2}{2!} + \ldots \frac{p(p-1)\ldots(p-n+1)}{n!} \ldots & & & |z| < 1 \\
\tan(z) & = & z + \frac{z^3}{3} + \frac{2z^5}{15} + \ldots & & & |z| < \frac{pi}{2}
\end{array}
$$

GROUPWORK

1. Write down the MacLaurin series for $f(z) = e^{1/z}$.

2. What is the value of $\text{Res}(e^{1/z}, 0)$?

3. Evaluate $\oint_{|z|=1} e^{1/z} dz$ **two different ways**.