# Complex Analysis

Math 312 MWF 10:30am - 11:25am
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*Class 28* **(Friday April 3)**

**SUMMARY** Poles, Zeroes, Residues and Cauchy's Residue Theorem
**CURRENT READING** Brown & Curchill pages 193-196
**NEXT READING** Brown & Curchill pages 180-196

**Zeroes and Poles**
So, we have had a lot of experience finding the **poles** of a function. The problem of finding a pole is equivalent to finding the **zero** of a related function. Let's formalize these definitions:
**Zero**
A point $z_0$ is called a **zero of order** $m$ for the function $f(z)$ if $f$ is analytic at $z_0$ and $f$ and its first $m-1$ derivatives vanish at $z_0$, but $f^{(m)}(z_0) \neq 0$.
**Pole**
A point $z_0$ is called a **pole of order** $m$ of $f(z)$ if $1/f$ has a zero of order $m$ at $z_0$.
**Identifying Poles and Zeroes**
Let $f$ be analytic. Then $f$ has a **zero of order** $m$at $z_0$ if and only if $f(z)$ can be written as $f(z) = g(z)(z-z_0)^m$ where $g$ is analytic at $z_0$ and $g(z_0) \neq 0$.

If $f(z)$ can be written as $f(z) = \dfrac{g(z)}{(z-z_0)^m}$ where $g(z)$ is analytic at $z_0$, then $f$ has a **pole of order** $m$ at $z = z_0$ and $g(z_0) \neq 0$

How do we find poles of a function? Well, if we have a quotient function $f(z) = p(z)/q(z)$ where $p(z)$ and $q(z)$ are analytic at $z_0$ and $p(z_0) \neq 0$ then $f(z)$ has a pole of order $m$ if and only if $q(z)$ has a zero of order $m$.
**Exercise**
We will classify all the singularities of $f(z) = \dfrac{3z+2}{z^2+1}$

**Groupwork**
Let's try and classify all the singularities of the following functions:

**(a)** $A(z) = \dfrac{4}{z^2(z-1)^3}$

**(b)** $B(z) = \dfrac{\sin z}{z^2-4}$

**(c)** $C(z) = \tan z$

**(d)** $D(z) = \dfrac{z}{z^2-6z+10}$

**Residues**

Once we know all the singularities of a function it is useful to compute the residues of that function. If a function $f(z)$ has a pole of order $m$ at $z_0$, the residue, denoted by **Res(**$f; z_0$) or **Res(**$z_0$**)** is given by the formula below:

$$\textbf{Res}(f; z_0) = \lim_{z \to z_0} \{ \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} [(z - z_0)^m f(z)] \}$$

Find the **residues** of all the singularities we previously classified.

**(a)**

**(b)**

**(c)**

**(d)**

**Cauchy's Residue Theorem**

If $f$ is analytic on a simple (positively oriented) closed contour and everywhere inside except the finite number of points $z_1, z_2, \cdots z_n$ inside , then

$$\oint f(z)\, dz = 2\pi i \sum_{k=1}^{n} \textbf{Res}(f; z_k)$$

Use Cauchy Residue Theorem to evaluate the following integrals:

**(a)** $\oint_{|z|=2} \dfrac{3z+2}{z^2+1}$

**(b)** $\oint_{|z|=5} \dfrac{4}{z^2(z-1)^3}\, dz$

**(c)** $\oint_{|z|=5\pi} \dfrac{\sin z}{z^2-4}\, dz$

**(d)** $\oint_{|z|=2\pi} \tan z\, dz$

**(e)** $\oint_{|z|=8} \dfrac{z}{z^2-6z+10}\, dz$