Complex Analysis

Math 312 Spring 1998 MWF 10:30am - 11:25am
Buckmire Fowler 112

Class 27 **(Wednesday April 1)**

SUMMARY Theorems Following Cauchy's Integral Formulas
CURRENT READING Brown & Curchill pages 130-135
NEXT READING Brown & Curchill pages 180-196

Implications of Cauchy's Integral Formula

We will take some time to see that we understand how the following theorems are derived from the CIF.

Cauchy's Inequality

Let f be analytic inside and on a circle $|z - z_0| = R$, taken in the positive sense, and denoted by C_R, then the derivatives of f as z_0 satisfy

$$|f^n(z_0)| \leq \frac{n!M}{R^n} \qquad (n = 1, 2, 3, \ldots)$$

where $|f(z)| \leq M$ for all z on C_R.

Proof: (Use the generalized CIF to try and bound $f^{(n)}(z_0)$)

Liouville's Theorem

If f is *entire* and *bounded* in the complex plane, then $f(z)$ is constant throughout the plane.

Proof: (Use Cauchy's Inequality when $n = 1$ and assume the radius has to be big enough to include the entire plane, e.g. $R \to \infty$)

Fundamental Theorem of Algebra

Every polynomial equation $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n = 0$ with degree $n \geq 1$ and $a_n \neq 0$ has atleast one root.

Proof: The proof is by contradiction and uses Liouville's Theorem.

Gauss' mean value theorem

If $f(z)$ is analytic inside and on a circle C with center z_0 and radius r then $f(z_0)$ is the mean of the values of $f(z)$ on C, namely

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta$$

Proof: This is obvious by using the CIF and a parametrization for $|z - z_0| = r$ and interpreting the results.

Maximum modulus theorem

If $f(z)$ is analytic inside and on a simple closed curve C and is not identically equal to a constant, then the maximum value of $|f(z)|$ occurs on C.

Proof:

Minimum modulus theorem

If $f(z)$ is analytic inside and on a simple closed curve C and $f(z) \neq 0$ inside C, then the minimum value of $|f(z)|$ occurs on C.

Proof: Use the maximum modulus theorem with $g(z) = 1/f(z)$.

Cauchy's Residue Theorem

If f is analytic on a simple (positively oriented) closed contour C and everywhere inside C except a finite number of points $z_1, z_2, \cdots z_n$ inside C, then

$$\oint_C f(z)\, dz = 2\pi i \sum_{k=1}^{n} \mathbf{Res}(z_k)$$

Before we talk about what a **residue** of a function $f(z)$ at z_k, denoted by $\text{Res}(f, z_k)$ or $\text{Res}(z_k)$ we will need to have a discussion about **poles**,**zeroes** and **singularities**.