# Complex Analysis

Math 312 Spring 1998 | MWF 10:30am - 11:25am
**Buckmire** | Fowler 112

*Class 27* **(Wednesday April 1)**

**SUMMARY** Theorems Following Cauchy's Integral Formulas
**CURRENT READING** Brown & Curchill pages 130-135
**NEXT READING** Brown & Curchill pages 180-196

**Implications of Cauchy's Integral Formula**

We will take some time to see that we understand how the following theorems are derived from the CIF.

**Cauchy's Inequality**

Let $f$ be analytic inside and on a circle $|z - z_0| = R$, taken in the positive sense, and denoted by $C_R$, then the derivatives of $f$ as $z_0$ satisfy

$$|f^n(z_0)| \leq \frac{n!M}{R^n} \qquad (n = 1, 2, 3, \ldots)$$

where $|f(z)| \leq M$ for all $z$ on $C_R$.

**Proof**: (Use the generalized CIF to try and bound $f^{(n)}(z_0)$)

**Liouville's Theorem**

If $f$ is *entire* and *bounded* in the complex plane, then $f(z)$ is constant throughout the plane.

**Proof**: (Use Cauchy's Inequality when $n = 1$ and assume the radius has to be big enough to include the entire plane, e.g. $R \to \infty$)

**Fundamental Theorem of Algebra**
Every polynomial equation $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n = 0$ with degree $n \geq 1$ and $a_n \neq 0$ has atleast one root.
**Proof**: The proof is by contradiction and uses Liouville's Theorem.

**Gauss' mean value theorem**
If $f(z)$ is analytic inside and on a circle $C$ with center $z_0$ and radius $r$ then $f(z_0)$ is the mean of the values of $f(z)$ on $C$, namely

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta$$

**Proof**: This is obvious by using the CIF and a parametrization for $|z - z_0| = r$ and interpreting the results.

**Maximum modulus theorem**
If $f(z)$ is analytic inside and on a simple closed curve $C$ and is not identically equal to a constant, then the maximum value of $|f(z)|$ occurs on $C$.
**Proof**:

**Minimum modulus theorem**
If $f(z)$ is analytic inside and on a simple closed curve $C$ and $f(z) \neq 0$ inside $C$, then the minimum value of $|f(z)|$ occurs on $C$.
**Proof**: Use the maximum modulus theorem with $g(z) = 1/f(z)$.

**Cauchy's Residue Theorem**
If $f$ is analytic on a simple (positively oriented) closed contour $C$ and everywhere inside $C$ except a finite number of points $z_1, z_2, \cdots z_n$ inside $C$, then

$$\oint_C f(z)\, dz = 2\pi i \sum_{k=1}^{n} \textbf{Res}(z_k)$$

Before we talk about what a **residue** of a function $f(z)$ at $z_k$, denoted by Res$(f, z_k)$ or Res$(z_k)$ we will need to have a discussion about **poles**,**zeroes** and **singularities**.