Complex Analysis

Class 23 **(Wednesday March 11)**

SUMMARY Applications of Cauchy-Goursat and simply-connected domains
CURRENT READING Brown & Curchill pages 104-111
NEXT READING Brown & Curchill pages 116-123

Previously we learned that the **Cauchy-Goursat Theorem** implies that,
given f is continuous in a domain D if any one of the following is true, then so are the others:

(a) f has an antiderivative in D, called $F(z)$, such that $F'(z) = f(z)$

(b) the integrals of $f(z)$ along contours lying entirely in D extending from z_1 to z_2 all have the same value, namely $F(z_2) - F(z_1)$

(c) the integrals of $f(z)$ around closed contours lying entirely in D all have value zero

Examples

Evaluate the following integrals. In each case sketch the contour you used and explain how you evaluated the integral, i.e. what idea are you using. Also answer the questions.

1. $\displaystyle\int_{-i}^{i} \frac{dz}{z}$

Does the integral value change depending on the path from $-i$ to i?
Does $1/z$ have an antiderivative? In what domain?

2. $\displaystyle\oint_{|z|=1} \frac{dz}{z}$

Does the integral value change depending on what direction one takes along $|z| = 1$?
What would the value of the integral be if the contour were traversed twice in the clockwise direction?

Loop

A **loop** or simple closed contour is said to be traversed in the **positive sense** if an observer traversing the contour in that direction would always see the *interior* of the contour on their left.

Simply-Connected and Multiply-Connected Domains
A **simply-connected** domain D is one such that *every* simple closed contour (i.e. loop) lying in D encloses only points of D

A domain which is *not* simply-connected, is called **multiply-connected**.

<u>GROUPWORK</u>
Sketch and classify the following domains as **simply-connected** or **multiply-connected**

(a) $|\text{Im } z| < 1$

(b) $1 < |z| < 2$

(c) $\mathbf{C} \setminus \{ \text{ Re}(z) > 0 \cap \text{ Im } (z) = 0 \}$

(d) $|z| < 4 \setminus \{|z - i| < .5 \cup |z + i| < .5\}$

FACT: Simply-connected domains have the property that every loop in D can be continuously deformed in D to a single point.

FACT: $\int_C f(z) dz \equiv 0$ if C is a point.

Deformation Invariance Theorem
Let f be an analytic function in a domain D containing loops , $_0$ and , $_1$. If these loops can be continuously deformed into each other by passing through points only in D, then

$$\oint_{\Gamma_0} f(z)\, dz = \oint_{\Gamma_1} f(z)\, dz$$

<u>**Examples**</u>
Evaluate the integral below, where , is some closed contour.

$$\oint_{\Gamma} \frac{dz}{z - z_0}$$

Does it matter where $z_0 \in \mathbf{C}$ is?