Complex Analysis

Math 312 Spring 1998 — MWF 10:30am - 11:25am
Buckmire — Fowler 112

Class 19 **(Monday March 2)**

SUMMARY Complex Functions of a **real** variable and complex contours
CURRENT READING Brown & Curchill, pages 86-94
NEXT READING Brown & Curchill pages 95-104

Previously we have considered complex functions of a complex variable, such as $w = f(z) = u(x,y) + iv(x,y)$
Now we want to consider complex functions which have a real variable as their argument. For example,

$$w(t) = u(t) + iv(t)$$

For the most part these functions just act like real functions. They consist of two real functions of one variable. They can be differentiated and integrated just like real functions.
Properties of Integrals of Complex Functions of a real variable

$$\begin{aligned} w'(t) &= u'(t) + iv'(t) \\ \int_a^b w(t)\, dt &= \int_a^b u(t)\, dt + i \int_a^b v(t)\, dt \\ \text{Re} \int_a^b w(t)\, dt &= \int_a^b \text{Re}(w(t))\, dt \\ \text{Im} \int_a^b w(t)\, dt &= \int_a^b \text{Im}(w(t))\, dt \\ \left| \int_a^b w(t)\, dt \right| &\leq \int_a^b |w(t)|\, dt \end{aligned}$$

Exercise
Consider $w_1(t) = 1 + it^2$ and $w_2 = e^{3it}$ Compute the following

1. $w_1'(t) =$
2. $w_2'(t) =$
3. $\int_0^2 w_1\, dt =$
4. $\int_0^2 w_2\, dt =$

Complex valued functions of a real variable are extremely useful in that they map a set of real points to a set of points in the complex plane.

Arcs

A point set $\gamma : z = (x, y)$ in the complex plane is said to be an **arc** if $x = x(t)$ and $y = y(t)$ where $a \leq t \leq b$, where $x(t)$ and $y(t)$ are continuous functions of t (which is real). The set γ is described by $z(t)$ where

$$z(t) = x(t) + iy(t), \qquad a \leq t \leq b$$

The arc γ is said to be a **simple arc** (also called a *Jordan arc*) if the arc never crosses itself. However, if the curve would be simple except that it crosses at the endpoints, i.e. $z(b) = z(a)$, it is called a **simple closed curve** or *Jordan curve*.

Length of an Arc

The length of an arc is given by

$$L = \int_a^b |z'(t)|dt = \int_a^b \sqrt{(x')^2 + (y')^2}dt$$

$x(t)$ and $y(t)$ can be thought of as parametric representations of the curve γ which consists of a set of points in the cartesian (x,y) plane.

GROUPWORK

(1) Give an example of a parametric representation of for the unit circle centered at (1,2). Sketch it below. (HINT: write the equation of the circle in Cartesian coordinates.)

(2) Draw a sketch of the points given by the parametricization

$z(t) = 3\sin(t) + i\cos(t), \qquad -2\pi \leq t \leq 2\pi$

(3) Use the information below to fully describe the curves you have just sketched.

Smooth Arcs

An arc is said to be **smooth** if it obeys the following three conditions

- $z(t)$ has a CONTINUOUS DERIVATIVE on the interval $[a, b]$
- $z'(t)$ is not zero on (a, b)
- $z(t)$ is a one-to-one function on $[a, b]$

If the first two conditions are met but $z(a) = z(b)$, then it is called a **smooth closed curve**.

Contours

A **contour** is a piecewise smooth arc. That is, $z(t)$ is continuous but $z'(t)$ is only piecewise continuous. If $z(a) = z(b)$ then it is called a *simple closed contour*.

Contours are important because they are the sets that **complex integration**, or integration of complex functions of a complex variable, are defined on.

Jordan curve theorem

A simple closed curve or simple closed contour divides the complex plane into two sets, the *interior* which is BOUNDED, and the *exterior*, which is UNBOUNDED.

This may seem obvious but is actually a very important insight into a feature of the set of points which make up the plane. You might try reading the proof suggested by the text to gain an appreciation for the non-trivial nature of the result.