Complex Analysis

Math 312 Spring 1998 MWF 10:30am - 11:25am
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Class 11 **(Friday February 6)**

SUMMARY Analyticity and Harmonic Functions
CURRENT READING Brown & Curchill, pages 59-64
NEXT READING Brown & Curchill pages 65-75

Update on ***Class 10***
We can summarize our knowledge of theorems on **analyticity**, **differentiability** and the **Cauchy-Riemann equations** with three statements:

ANALYTICITY $\iff$ Existence of $f'(z)$
ANALYTICITY $\Rightarrow$ C.R.E.
ANALYTICITY $\iff$ C.R.E. **AND** Continuity of u_x, u_y, v_x, v_y

Therefore if we want to show a given function is **not analytic** "at a point," we can either show that the CRE are not satisified, or that the derivative does not exist at that point.
We easily showed that $f(z) = \overline{z} = x - iy$ did **not** satisfy the CRE. However we had some problem showing the derivative $f'(z)$ did not exist. This is because I chose two vertical paths to take limits on. However, if we choose a vertical path to take the limit we get the results that $f'(z) = -1$ (for all z).
However, if we choose a horizontal path, it is easy to show (**can YOU?**) that $f'(z) = +1$ (for all z). Thus, since $1 \neq -1$, the derivative of $f(z) = \overline{z}$ does not exist for any z, so $f(z) = \overline{z}$ is not analytic anywhere.

Laplace's Equation
The partial differential equation shown below is known as **Laplace's Equation**.

$$\nabla^2 \phi = \Delta \phi = \frac{\partial^2 \phi(x,y)}{\partial x^2} + \frac{\partial^2 \phi(x,y)}{\partial y^2} = 0$$

Solutions $\phi(x, y)$ which solve Laplace's equation are very important in a number of areas of mathematical physics and applied mathematics. Some of these applications are:

- electrostatic potential in two-dimensional free space
- scalar magnetostatic potential
- stream function and velocity potential in fluid flow (aerodynamics, etc)
- spatial distribution of equilibrium temperature

Harmonic Functions

A real-valued function $\phi(x,y)$ is said to be **harmonic** in a domain (i.e. open, connected set) D if all its second-order partial derivatives are continuous in D and if ϕ satisfies Laplace's Equation at each point $(x,y) \in D$.

If $f(z)$ is analytic on a domain D then both $u(x,y) =$Re$(f(z))$ and $v(x,y) =$Im$(f(z))$ are harmonic in D.

ANALYTICITY $\Longleftrightarrow$ Re $f(z)$ and Im $f(z)$ are HARMONIC

The proof follows directly from the CRE.

(Take 3 minutes and try and come up with it.)

Given a harmonic function $u(x,y)$ defined on an open connected set D we can construct a **harmonic conjugate** $v(x,y)$ so that the combined function $f = u(x,y) + iv(x,y)$ will be analytic on the domain D.

Example

Given $u(x,y) = x^3 - 3xy^2 + y$ find the harmonic conjugate $v(x,y)$ and thus construct an analytic function $f(z)$ such that Re $f(z) = u(x,y)$

By studying harmonic functions we can learn about analytic functions, and vice-versa. Harmonic functions also appear in the analysis of a number of physical phenomena.

The Complex Velocity Potential

In Fluid Dynamics, the complex velocity potential is a useful quantity used to analyze certain fluid fields. It can be defined as

$$\boldsymbol{\Phi(z) = \phi(x,y) + i\psi(x,y)}$$

where $\phi(x,y)$ is called the *velocity potential* and $\psi(x,y)$ is called the *stream function*.

One can compute expressions for V_x (horizontal component) and V_y (vertical component) of the velocity of a fluid, denoted by $\vec{\boldsymbol{v}}$ from the complex velocity potential:

$$\boldsymbol{\frac{\partial \overline{\Phi}}{\partial z} = V_x + iV_y = \frac{\partial \phi}{\partial x} + i\frac{\partial \phi}{\partial y} = \vec{v}}$$

Example

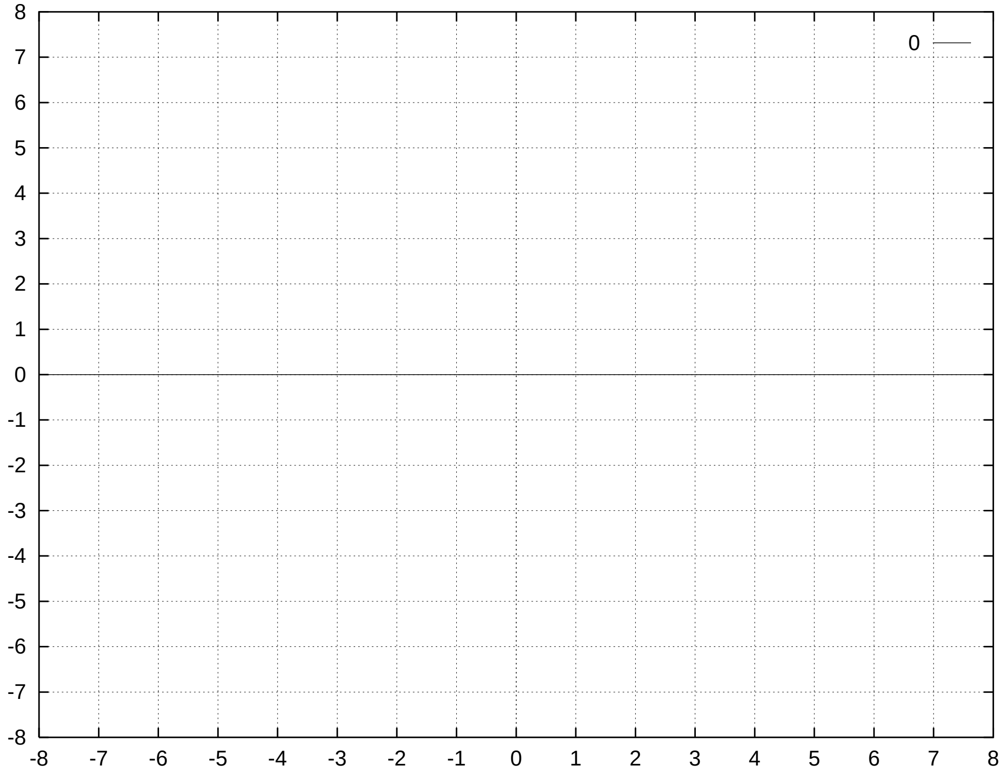
Given $\boldsymbol{\Phi(z) = z^2}$ compute the stream function ψ and velocity potential ϕ and obtain expressions for fluid velocity $\vec{\boldsymbol{v}}(\boldsymbol{x,y})$

Streamlines and Equipotentials

Recall that the level curves of a function $f(x, y)$ occur when $f(x, y) = \text{constant}$. Level curves of $\phi(x, y)$ are called **equipotentials** and level curves of $\psi(x, y)$ are called **streamlines**. They have particularly interesting physical meaning.

GROUPWORK

Sketch the streamlines and equipotentials relating to the flow described by $\Phi(\boldsymbol{z}) = \boldsymbol{z}^2$ on the grid below. In other words, sketch $\phi(x, y) = c$ and $\psi = d$, where c and d are $\pm 1, \pm 2$, et cetera. What kind of curves are they?



If you look carefully (or if you sketched accurately) the equipotentials and streamlines intersect at right angles. This is not an accident. Level curves for the real and imaginary parts of an analytic function $f(z)$ are always **orthogonal**.

We can show this by remembering the meaning of the gradient of a function $f(x, y)$, denoted by ∇f, the dot product and applying the Cauchy-Riemann equations:

Example

Show that the level curves of harmonic conjugates of an analytic function always intersect perpendicularly.