Complex Analysis

Math 312 Spring 1998 — MWF 10:30am - 11:25am
Buckmire — Fowler 112

Worksheet #3 (Friday 01 / 16 / 98)

SUMMARY Polar and Exponential forms of Complex Numbers

READING Churchill & Brown, pages 12-18

We have been considering the complex plane as an analogue to the 2-D cartesian x-y plane. You may recall that there are other coordinate systems that can be imposed on the 2-D plane. One of those coordinate systems is known as **polar coordinates**.

Exercise

Write $(1, \sqrt{3})$ in polar coordinates.

What is the angle between a line drawn from $(0, 0)$ to $(1, \sqrt{3})$ and the x-axis?

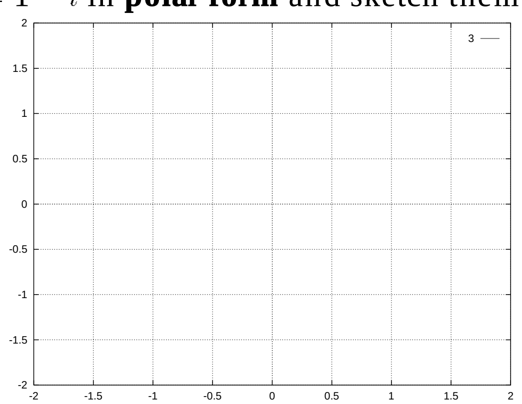
The complex number which is found at $(1, \sqrt{3})$ is ____________.
It can also be written as $z = r(\cos\theta + i\sin\theta) = |z| \text{ cis } \theta$
This is known as the **polar form** of the complex number.
This angle θ corresponding to the complex number z is called the **principal argument** of z, denoted by $\text{Arg } z$. By tradition, $-\pi < \text{Arg } z \le \pi$ and Arg 0 is undefined.
The set of all values θ of such that $\tan\theta = \dfrac{\text{Im } z}{\text{Re } z}$ is known as the **argument** of z, or $\arg z$

$$\arg z = \text{Arg } z + 2n\pi \qquad (n = 0, \pm 1, \pm 2, \ldots) \tag{1}$$

Exercise

Therefore, $\arg(1 + \sqrt{3}i)$ is ______________________________.

Write $z_1 = -1 - i$ and $z_2 = 1 - i$ in **polar form** and sketch them on the graph below.



Do you see any interesting geometrical relationship between z_1 and z_2?

Do you notice any interesting algebraic relationship between z_1 and z_2?

Write down a general relationship between $\arg(z_1 z_2)$ and $\arg(z_1)$ and $\arg(z_2)$

Exponential Forms

Euler's Formula

You may have noticed that the expression $\cos\theta + i\sin\theta$ is so common that the shorthand $\text{cis}\,\theta$ has been developed. Another, even more compact symbol can be derived using Euler's formula:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Note $\theta = \arg z$ and is a real number in these formulas. So, now we can write complex numbers in **exponential form**

$$z = |z|e^{i\theta} = re^{i\theta}$$

Exercise

1. Compute the value of $e^{\pi i}$

2. Write $1 + i$ in exponential form

3. Write $(1 - i)^5$ in rectangular form (i.e. the usual $x + iy$)

De Moivre's Formula

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

We will prove this is true by first writing down z^n in polar form and exponential form and equating them.