Complex Analysis

Math 312 Spring 1998 — MWF 10:30am - 11:25am
Buckmire — Fowler 112

Homework Set 6

45 + 10 bonus points

ASSIGNED: Fri April 03 1998 — **DUE: Fri April 10 1998**

1. (*2 points*) Find the maximum and minimum values of $|z^2 + 2z - 1|$ in the disk $|z| \leq 1$

2. (*3 points*) Suppose $f(z) = 1/(1-z)^2$, and $0 < R < 1$.

   (a) Show that $\max_{|z|=R} |f(z)| = 1/(1-R^2)$

   (b) Show that $f^{(n)}(0) = (n+1)!$

   (c) Use Cauchy's Inequality to show that $(n+1)! \leq \dfrac{n!}{R^n(1-R)^2}$

3. *(5 points)* **#1, page 188.** Find the residue of each function $f(z)$ at $z = 0$

   (a) $f(z) = \dfrac{1}{z^2+z}$ **(b)** $f(z) = z\cos\left(\frac{1}{z}\right)$ **(c)** $f(z) = \dfrac{\cot z}{z^4}$

   (d) $f(z) = \dfrac{\sinh z}{z^4(1-z^2)}$ **(e)** $f(z) = \dfrac{z - \sin z}{z}$

4. (*4 points*) **#2, page 189.** Use Theorem 1 on page 183 *(Cauchy's Residue Theorem*) to evaluate the integral of each of these functions around the circle $|z| = 3$ in the positive sense: **(a)** $f(z) = \dfrac{\exp(-z)}{z^2}$ **(b)** $f(z) = \dfrac{\exp(-z)}{(z-1)^2}$ **(c)** $f(z) = z^2 \exp\left(\dfrac{1}{z}\right)$ **(d)** $f(z) = \dfrac{z+1}{z^2-2z}$

5. (*3 points*) **#3, page 189.** Use Theorem 2 on page 183 (*"Cauchy's Second Residue Theorem"*) to evaluate the integral of each of these functions around the circle $|z| = 2$ in the positive sense:

   (a) $f(z) = \dfrac{z^5}{1-z^3}$ **(b)** $f(z) = \dfrac{1)}{1+z^2}$ **(c)** $f(z) = \dfrac{1}{z}$

6. (*5 points*) **#4, page 189.** In each case, write the *principal part* of the function at its isolated singular point and determine whether that point is a pole, removable singular point, or an essential singular point:

   (a) $z\exp\left(\dfrac{1}{z}\right)$ **(b)** $\dfrac{z^2}{1+z^2}$ **(c)** $\dfrac{\sin(z)}{z}$ **(d)** $\dfrac{\cos z}{z}$ **(e)** $\dfrac{1}{(2-z)^3}$

7. (*3 points*) **#5, page 189.** Show that the singular point of each function $f(z)$ is a pole. Determine the order m of the pole and the corresponding residue B.

   (a) $f(z) = \dfrac{1-\cosh(z)}{z^3}$ **(b)** $f(z) = \dfrac{1-\exp(2z)}{z^4}$ **(c)** $f(z) = \dfrac{\exp(2z)}{(z-1)^2}$

8. (*3 points*) **#4, page 197.** Find the value of the integral $\displaystyle\oint_C \frac{3z^3+2}{(z-1)(z^2+9)}dz$ taken clockwise around the circle **(a)** $|z-2| = 2$ and **(b)** $|z| = 4$

9. (*3 points*) **#5, page 197.** Find the value of the integral $\oint_C \frac{dz}{z^3(z+4)}$ taken clockwise around the circle **(a)** $|z| = 2$ and **(b)** $|z+2| = 3$

10. (*3 points*) **#6, page 197.** Let C be the circle $|z| = 2$ described in the positive sense, and evaluate the integral **(a)** $\oint_C \tan(z)\, dz$ **(b)** $\frac{dz}{\sinh(2z)}$ **(c)** $\frac{\cosh(\pi z)}{(z^2+1)z}$

11. (*6 points*) **#7, page 197.** Use *Cauchy's Second Residue Theorem* (found on page 185) to evaluate integrals of the following functions around the positively oriented circle $|z| = 3$ by computing only one residue in each case. **(a)** $f(z) = \frac{(3z+2)^2}{z(z-1)(2z+5)}$ **(b)** $f(z) = \frac{z^3(1-3z)}{(1+2z^4)(1+z)}$ **(c)** $f(z) = \frac{z^3 e^{1/z}}{1+z^3}$

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(*5 points*) Use a separate sheet of paper to discuss your understanding of residues and singularities. What is a residue and why is it useful to compute it? How do **you** evaluate it? How many different types of singularity do you know? What's the difference between the different types? Comment on the relationship between singularities, residues and contour integrals. You should select a particular problem from this homework set to discuss how these three concepts are linked. Write atleast three paragraphs.

BONUS (*10 points*) **#8 page 197-98.** Let C_N denote the positively oriented boundary of the square whose edges lie along the lines

$$x = \pm\left(N + \frac{1}{2}\right)\pi, \qquad \text{and} \qquad \pm\left(N + \frac{1}{2}\right)\pi,$$

where N is a positive integer. Show that

$$\oint_{C_N} \frac{dz}{z^2 \sin(z)} = 2\pi i \left[\frac{1}{6} + 2\sum_{n=1} N \frac{(-1)^n}{n^2\pi^2}\right]$$

Then, using the fact that the value of this integral tends to zero as N tends to infinity (**Why?**) point out how it follows that

$$\sum_1^\infty \frac{(-1)^{n+1}}{n^2} = fracpi^2 12$$

NOTES

This homework sets is due in class on **Friday April 10**. You are **strongly** encouraged to work collaboratively on the homework, though each person must hand in indvidually-written work. You should indicate on your neatly-written solution manuscripts which students you collaborated with. If you encounter difficulty, you should ask questions on the Complex Analysis wwwboard, the Complex Analysis email list, or come see me in my office (during Office Hours Wed 1-5 and Thu 3-5 or schedule an appointment).