Complex Analysis

Math 312 Spring 1998 | MWF 10:30am - 11:25am
Buckmire | Fowler 112

Homework Set 4

ASSIGNED: Wed March 4 1998 | **DUE: Fri March 13 1998**

1. (*2 pts*) Find a linear transformation which maps the circle $|z+2|=1$ onto the circle $|w-5|=3$

2. (*3 pts*) Find a function which maps the upper-half plane to the disk $|2w-i+1| \leq 1$

3. (*10 pts*) Map $|z-2|=1$ under the following mappings (sketch the results). Notice all of these mappings are LFTs of the form $w = \dfrac{az+b}{cz+d}$. For each mapping write down the corresponding values of the parameters a, b, c and d.
   (a) $w = z - 2i$ **(b)** $w = 3iz$ **(c)** $w = 4z + 1 + i$
   (d) $w = \dfrac{z-2}{z-1}$ **(e)** $w = \dfrac{z-4}{z-3}$ **(f)** $w = \dfrac{1}{z}$

4. (*4 pts*) Find the Mӧbius transformation which maps $0, 1, \infty$ to the following respective points: **(a)** $0, i, \infty$ **(b)** $0, 1, 2$ **(c)** $-i, \infty, 1$ **(d)** $-1, \infty, 1$ After finding $w = T(z)$ sketch the pre-image and image boundaries in the z-plane and $w - plane$.

5. *(3 pts)* What is the image of the third quadrant under $w = \dfrac{z+i}{z-i}$?

6. (*3 pts*) Find an LFT which takes the half-plane $x - y < 1$ to unit disk $|w| < 1$

BONUS:
(*10 points*) Find $M(z)$ which maps the shaded region on the left ($|z| \leq 2 \cap |z-1| \geq 1$) onto the shaded region on the right ($|w-1| \leq 1$).

[HINT: Use an LFT which maps both circular boundaries to parallel, horizontal lines. That is, your LFT will map 2 to ∞ and convert the shaded area between circles to a shaded area between lines (a horizontal strip). Then use the exponential mapping to map that strip to the upper-half plane. Then map the upper half-plane to the interior of the unit circle and shift.]

The following problems are found on page 102 in the text *Brown & Churchill*: **1, 3, 4, 5, 6, 9, 15**. Most of the answers to these problems are given. Therefore you must be more diligent than usual in explaining your solutions in order to get full credit. For **#1, #3 – #6** use parametric representations for C in order to evaluate $\int_C f(z)dz$.

7. (*3 pts*) **#1, p 102** $f(z) = (z+2)/z$ and C is

   (a) the semi-circle $z = 2e^{i\theta} \qquad (0 \le \theta \le \pi)$

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8. (*2 pts*) **#3, p 102** $f(z) = \pi \exp(\pi\overline{z})$ and C is the boundary of the square with vertices at the points $0, 1, 1+i$ and i, with the orientation of C being counterclockwise.

9. *(2 pts)* **#4, p 102** $f(z)$ is defined by the equations

$$f(z) = \begin{cases} 1, & \text{when} \quad y < 0 \\ 4y, & \text{when} \quad y > 0 \end{cases}$$

   and C is the arc from $z = -1 - i$ to $z = 1 + i$ along the curve $y = x^3$.

10. (*2 pts*) **#5, p 102** $f(z) = 1$ and C is an *arbitrary* contour from a point z_1 to another point z_2 in the plane.

11. (*4 pts*) **#6, p. 102** $f(z)$ is the branch

$$z^{-1+i} = \exp((-1+i)\log z) \quad (|z| > 0,\ 0 < \arg z < 2\pi) = \exp((-1+i)\mathcal{L}_0(z))$$

    of the power function z^{-1+i}, and C is the positively-oriented unit circle $|z| = 1$.

12. (*3 pts*) **#9, p. 103** Let C be the arc of the circle $|z| = 2$ from $z = 2$ to $z = 2i$ that lies in the first quadrant. Without evaluating the integral, show that $\left| \int_C \frac{dz}{z^2 - 1} \right| \le \frac{\pi}{3}$

13. (*4 pts*) **#15, p 103** Let C_R be the circle $|z| = R(R > 1)$ described in the counterclockwise direction. Show that $\left| \int_C \frac{\operatorname{Log} z}{z^2} \right| \le 2\pi \frac{\pi + \ln R}{R}$ and then use L'Hospital's Rule to show that as $R \to \infty$ the value of the integral becomes zero.

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(*5 points*) Use a separate sheet of paper to discuss the mapping process. How do you approach problems which involve mappings? What difficulties do you have? Give an example of a problem where you are given a mapping and a pre-image and are asked to find the image *and* and example where you are asked to find a mapping which transforms a given pre-image to a given image. Discuss how you solve these problems differently. Write atleast three paragraphs.

NOTES

This homework sets is due in class on **Friday February 13**. You are **strongly** encouraged to work collaboratively on the homework, though each person must hand in indvidually-written work. You should indicate on your neatly-written solution manuscripts which students you collaborated with. If you encounter difficulty, you should ask questions on the Complex Analysis wwwboard, the Complex Analysis email list, or come see me in my office (during Office Hours Wed 1-5 and Thu 3-5 or schedule an appointment).