Complex Analysis

Math 312 Spring 1998 — MWF 10:30am - 11:25am

Buckmire — Fowler 112

Homework Set 2

ASSIGNED: Friday Feb 6 — **DUE: Friday Feb 13**

1. Write each of the following functions in the form $f(z) = u(x, y) + iv(x, y)$

   (a) $f(z) = z^3 + iz + 1$

   (b) $f(z) = \dfrac{1}{z}$

   (c) $f(z) = \dfrac{z+i}{z^2+1}$

2. Compute $f'(z)$ for each function in the previous question

3. Consider $f(z) = \dfrac{2z+1}{3z-2}$. Write down

   (a) The domain of definition of f

   (b) The range of f

   (c) $f(f(z))$

   (d) $f\left(\dfrac{1}{z}\right)$

   (e) $\lim_{z\to 0} f(z)$

   (f) $\lim_{z\to\infty} f(z)$

4. Prove that $f(z) = \bar{z}$ is continuous everywhere in the complex plane.

5. Given $F(z) = z + i$, $G(z) = e^{\frac{i\pi}{4}}$ and $H(z) = z/2$, sketch the image (and describe it in terms of complex inequality) of applying the following composite mappings on the semi-disk $|z| \leq 2 \cap \text{ Im } z \geq 0$

   (a) $F(z)$

   (b) $G(z)$

   (c) $H(z)$

   (d) $F(G(z))$

   (e) $G(H(z))$

   (f) $H(F(z))$

   (g) $F(G(H(z)))$

6. Prove that $\text{Re } z$, $\text{Im } z$ and $|z|^2$ are nowhere differentiable.

7. The mapping $w = g(z) = 1/z$ is called an **inversion mapping**.

   (a) Show that the circle $|z| = r$ gets mapped onto the circle $|w| = 1/r$

   (b) Show that the ray Arg $z = \alpha, -\pi < \alpha\pi$, onto the ray Arg $w = -\alpha$

8. Find the range of each of the following functions:

   (a) $g(z) = z^2$ for Re $z \geq 0 \cap$ Im $z \geq 0$

   (b) $h(z) = z + 5$ for Re $z > 0$

9. Given the complex potential $\Phi(\boldsymbol{z}) = (\cos \boldsymbol{\alpha} - \boldsymbol{i} \sin \boldsymbol{\alpha})\boldsymbol{z}$

   (a) Sketch the equipotentials and give equations for them

   (b) Sketch the streamlines and give equations for them

   (c) Find the components V_x and V_y of the velocity vector $\vec{v}$

   (d) What angle does the vector $\vec{v}$ make with the x-axis?

10. Determine for what values of z the derivative $f'(z)$ exists:

    (a) $f(z) = z$ Im z

    (b) $f(z) = x^3 + i(1 - y)^3$

11. Show that if $f(z)$ is analytic then $f''(z) = u_{xx} + iv_{xx} = -u_{yy} - iv_{yy}$

12. Find the most general form of $\phi(x, y) = ax^2 + bxy + cy^2$ which is harmonic.

NOTES

All homework sets will be due in class *at least* one week from the class they are given out in. You are strongly encouraged to work together on the homework and to come see me for help with any of the problems. Please indicate on your answer sheets the names of the other students with whom you collaborated.