Complex Analysis

Math 312 MWF 10:30-11:25am
Spring 1996 201 Fowler

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Worksheet #30

SUMMARY Poles, Zeroes, Residues and Cauchy's Residue Theorem

READ page 130 (Section 43)

Zeroes and Poles
So, we have had a lot of experience finding the **poles** of a function. The problem of finding a pole is equivalent to finding the **zero** of a related function. Let's formalize these definitions:
Zeroes
A point z_0 is called a **zero of order** m for the function $f(z)$ if f is analytic at z_0 and f and it's first $m-1$ derivatives vanish at z_0, but $f^{(m)}(z_0) \neq 0$.
Pole
A point z_0 is called a **pole of order** m of $f(z)$ if $1/f$ has a zero of order m at z_0.
Identifying Poles and Zeroes
Let f be analytic. f has a **zero of order** $m$at z_0 if and only if $f(z)$ can be written as $f(z) = g(z)(z-z_0)^m$ where g is analytic at z_0 and $g(z_0) \neq 0$.

If $f(z)$ can be written as $f(z) = \dfrac{g(z)}{(z-z_0)^m}$ where $g(z)$ is analytic at z_0, then f has a **pole of order** m at $z = z_0$ and $g(z_0) \neq 0$

How do we find poles of a function? Well, if we have a quotient function $f(z) = p(z)/q(z)$ where $p(z)$ and $q(z)$ are analytic at z_0 and $p(z_0) \neq 0$ then $f(z)$ has a pole of order m if and only if $q(z)$ has a zero of order m.

Examples
Let's try and classify all the singularities of the following functions:

(a) $\dfrac{\cos z}{z^2(z-\pi)^3}$

(b) $\dfrac{\sin z}{z^2-4}$

(c) $\tan z$

(d) $\dfrac{z}{z^2-6z+10}$

(e) $\dfrac{3z+2}{z^4+1}$

Residues

Once we know all the singularities of a function it is useful to compute the residues of that function. If a function $f(z)$ has a pole of order m at z_0, the residue, denoted by $\mathbf{Res}(f; z_0)$ or $\mathbf{Res}(z_0)$ is given by the formula below:

$$\mathbf{Res}(f; z_0) = \lim_{z \to z_0} \{ \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} [(z - z_0)^m f(z)] \}$$

Find the **residues** of all the singularities we previously classified.

Cauchy's Residue Theorem

If f is analytic on a simple (positively oriented) closed contour Γ and everywhere inside Γ except the finite number of points $z_1, z_2, \cdots z_n$ inside Γ, then

$$\oint_\Gamma f(z)\, dz = 2\pi i \sum_{k=1}^{n} \mathbf{Res}(z_k)$$

Use Cauchy Residue Theorem to evaluate the following integrals:

(a) $\displaystyle\oint_{|z|=5} \frac{\cos z}{z^2 (z - \pi)^3}\, dz$

(b) $\displaystyle\oint_{|z|=5\pi} \frac{\sin z}{z^2 - 4}\, dz$

(c) $\displaystyle\oint_{|z|=2\pi} \tan z\, dz$

(d) $\displaystyle\oint_{|z|=8} \frac{z}{z^2 - 6z + 10}\, dz$

(e) $\displaystyle\oint_{|z|=2} \frac{3z + 2}{z^2 + 1}$